Matching through Institutions

Francis Bloch[a], David Cantala[b], Damián Gibaja[c]

[a] *Université Paris 1 and Paris School of Economics, francis.bloch@univ-paris1.fr*
[b] *El Colegio de México, dcantala@colmex.mx*
[c] *Universidad Popular Autónoma del Estado de Puebla, damianemilio.gibaja@upaep.mx*

Abstract

We analyze a three-sided matching market where institutions own objects and individuals belong to institutions. Institutions pool their objects to enlarge the choice set of individuals. For any institution, the number of individuals who receive an object must be equal to the number of objects initially owned. Under this distributional constraint, individually rational and fair assignments may fail to exist. However, when the number of individuals is sufficiently large, fair assignments exist and can be found using a new algorithm, called the Nested Deferred Acceptance algorithm with interrupters (NDAI). This procedure nests a one-to-one matching between agents and objects and a one-to-many matching between objects and institutions. We show that it outputs a matching which is Pareto optimal among fair matchings and strategy-proof for individuals. When agents belong to several institutions, the NDAI results in assignments which are fair for agents of the same institution.

Keywords: Matching, Institutions, Deferred Acceptance Algorithm, Social Housing

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1. Introduction

This paper studies matching markets intermediated by institutions. Institutions own objects (or seats or slots) and agents are attached to institutions. In autarky, institutions assign their objets to agents who belong to the institution. When pooling is allowed, institutions can exchange objects in order to expand the choice set of all agents. Our objective is to study these flexible assignment rules when (i) agents have preferences over the entire set of objects, (ii) an institution has an absolute priority over the objects it owns, but other institutions may also access the object according to a fixed priority rule and (iii) institutions have preferences over the assignment of objects to agents who belong to the institution. We thus model matching markets as three-sided markets, where institutions exchange objects and objects are assigned to agents. Distributional constraints imply that, in the end, each institution must equalize the number of objects it owns with the number of agents on its list who receive an object.

Matching through institutions occurs in many different situations. We first encountered it in the context of the allocation of social housing in Paris. Apartments are owned by four different institutions (the French state, the city of Paris, local councils and private firms). The centralized allocation procedure runs as follows. First, each institution receives a fixed quota of apartments corresponding to the number of vacant apartments they bring to the pool. Then households report the apartments they want to rent. The assignment is decided by a committee involving all institutions. It allocates apartments to households, under the constraint that every institution equalizes its number of apartments with the number of households on its list who receive an apartment.[1] Different institutions pursue different objectives. The French state seeks to provide emergency housing for battered women or refugees, the city of Paris wants to promote a balance between lower income and higher income residents in the inner city, local councils and private firms primarily want to house their own employees. Hence, objectives of institutions may differ from household preferences and vary from institution to institution.

Balanced exchange programs, such as tuition and student exchange programs, provide another example of matching mechanisms where institutions pool objects. Tuition exchange programs allow children of staff from one university to benefit from reduced tuition in other universities.[2] Children of staff are usually offered free or reduced tuition in their own universities. Through tuition exchange programs, universities pool their resources so that children of staff can benefit from reduced tuition in a larger set of universities. In order to maintain financial balance, the number of incoming and outgoing students in each university must be equalized.

Numerous student-exchange programs exist, both at the university level and at the level of secondary schools. The Erasmus program is a thriving program of student exchanges, involving almost all European universities. Students from one university can study in another university with no financial competition, as long as a balance between incoming and outgoing students is maintained. American universities also

[1]For a detailed description of the mechanism, see the annual report on the assignment of social housing in Paris [1].
[2]See Dur and Uñver [2] for a description of these programs.

organize student exchanges for study-abroad programs. Because all universities cannot be present in all countries, they often pool resources, allowing students from other colleges and universities to participate in their programs. These agreements are sometimes based on transfer payment and sometimes made on a quid pro quo basis, where the number of incoming and outgoing students are matched for each university.[3] At the secondary education level, inter-district school choice programs, like the one sponsored by the state of New Jersey, [4] also expand the choice set of students, These programs allow school districts to exchange students and to specialize in specific programs, such as arts programs or programs for students with special needs. Pupils from one school district can be assigned to a school in another district up to a limit in the number of seats. Inter-district school programs are either based on financial transfers across school districts or, as in the case of New Jersey, on state funding, which sets a priority for each seat over school districts.

We model matching through institutions as a three-sided market involving agents (which we refer to as "households"), objects (which we refer to as "apartments"), and owners (which we refer to as "institutions"). We first show that distributional constraints (implying that the number of apartments owned by each institution is equal to the number of agents from the institution who receive an apartment) results in two major complications. First, it may lead to the nonexistence of feasible assignments. Second, even when feasible assignments exist, they may be subject to justified envy. In order to guarantee existence of feasible assignments, we assume that each institution/agent has a sufficiently long list of acceptable agents/apartments so that every apartment is acceptable to some agent in the list. Under this sufficient condition –termed the over-demand condition– we show that fair assignments satisfying distributional constraints exist.

Our objective is then to produce assignments that satisfy the classical properties of individual rationality, non-wastefulness, fairness, and strategy-proofness, adapted to this new setting where apartments are a common pool of resources. The main contribution of the paper is to propose a new algorithm, the Nested Deferred Acceptance (NDA) algorithm, which nests two different deferred acceptance algorithms that interact in parallel. In the first one (the "outer loop"), each household asks for her most preferred apartment among those that have not yet rejected her. Given this list of demands, we run a second deferred acceptance algorithm (the "inner loop") among institutions. Each institution chooses a set of household-apartment pairs that maximizes its preferences and does not exceed its vector of quotas. If more than one institution asks for one apartment, it is assigned according to the priority of each apartment. Hence, the inner loop may assign apartments to institutions that do not initially own them. The process is repeated until apartments are assigned to households in such a way that institutions respect their quotas. Going back to the outer loop, we next ask rejected households to apply for their next preferred apartment, and the procedure continues until no household is rejected. If institutions do not

[3]In most cases, the agreements among colleges are bilateral. In some cases, like in the University of California system, there exists a central clearinghouse assigning students to the programs offered by different universities.

[4]See www.nj.gov/education/choice for a description of the system.

share apartments, this algorithm is equal to the Students Optimal Stable Mechanism (SOSM).

We first show that individuals always benefit from the enlarged choice set when apartments are assigned according to the NDA algorithm. We also show that, in the absence of distributional constraints, the classical theory on two-sided matching naturally extends to our three-sided problem. The NDA mechanism is strategy-proof for households and outputs a fair assignment that is Pareto undominated by any fair assignment. Distributional constraints imply that the NDA may not produce a fair assignment. We observe that justified envy arises because of the presence of interruptions caused by institutions. To restore fairness, we introduce a modified version of the NDA algorithm that identifies and deletes "interrupters", i.e. institutions which are temporarily assigned to apartments that they will not be assigned to in the final matching, thereby blocking access of an apartment to households from other institutions. The problem arises because institutions do not demand household-apartments pairs in the decreasing order of their preferences, but are constrained by the demand expressed by households. The phenomenon is thus different from the "interrupters" defined in Kesten [3] in the context of school choice, where interrupters need to be deleted to restore Pareto efficiency rather than fairness. The solution to both types of interruptions, however, is identical. As in Kesten [3], we solve the problem by modifying the preferences of interrupters, deleting apartments from their preferences, and resulting in a modified NDA algorithm called the Nested Deferred Acceptance with Interrupters (NDAI) algorithm. Our main result shows that the output given by the NDAI is fair, Pareto undominated by any other fair assignment satisfying the distributional constraints and strategy-proof.

We also analyze the case where the same agent can appear on the list of multiple institutions (This may be the case in the assignment of social housing in Paris). We show that fair assignments do not necessarily exist and that the NDAI then produces an assignment which is not fair but only fair among agents belonging to the same institution. Finally, while we deal with strict quotas in the paper, we show that the NDA also applies to situations with maximum quotas, as in Kamada and Kojima [4]'s model of flexible assignment of doctors to hospitals in Japan. In an Appendix, we show that the NDA algorithm can be adapted to matching with distributional constraints, by reinterpreting regions as "institutions," doctors as "households" and jobs in hospitals as "apartments." The NDA, however, does not adapt to general distributional constraints, like Goto et al. [5].

1.1. Relation with the literature

Three-sided markets. The paper is related to the literature of three-sided markets. This literature discusses conditions for existence of stable matchings (Alkan [6] and Huang [7]), focusing on different preferences domains: cyclic preferences (Biró and McDermid [8]), binary relations (Farczadi, Georgiu and Köneman [9]), and hybrid preferences (Zhang and et al. [10]). We contribute to this literature by providing a setting where it is possible to find a stable (fair) matching by using the NDA, which is an algorithm that runs in polynomial time.

Matching with constraints. The problem we consider is related to the rapidly emerging literature on matchings with constraints. In the school choice context, a number of recent papers have analyzed the effect of constraints resulting from affirmative action considerations. One stream of papers interpret affirmative action as "leveling the playing field," as in Kojima [11], Hafalir, Yenmez and Yildirim [12] and Kominers and Sönmez [13]. Another stream of papers closer to our motivation consider affirmative action is an objective per se, formalized either by the existence of quotas as in Abdulkadiroğlu [14], Abdulkadiroğlu and Sönmez [15] and Hafalir, Yenmez and Yildirim [12], or bounds as in Ehlers, Hafalir, Yenmez, and Yildirim [16], Fragiadakis and Troyan [17] and Bó [18]. One of the objectives of these papers is to refine stability concepts and the deferred acceptance algorithms to conform to the bounds and quotas. Echenique and Yenmez [19], Erdil and Kumano [20], and Biró, Klijn and Pápai [21] consider diversity as an objective of the school district and explore the tension between diversity objectives, stability, and efficiency of priority systems and matching rules. Nguyen and Vohra [22] have a different approach and use Scarf's Lemma to implement proportional distributional constraints.

Balanced matching. Our analysis also bears a close connection to exchange markets with balanced constraints recently studied by Dur and Uṅver [2], Biró, Klijn, and Pápai [21] and Hafalir, Kojima and Yenmez [23]. These papers model exchange programs (like the tuition exchange for children of faculty members or the Erasmus exchange program in European universities) where a balance must be kept between the number of incoming and outgoing students. The papers consider a two-sided (rather than three-sided) matching problem, where colleges have preferences over students, and students have preferences over colleges. Contrary to us, they do not model preferences of institutions over the assignment of students to colleges. In the tuition exchange model of Dur and Uṅver [2], students are ranked inside each college according to an exogenous priority (for example, the length of tenure of a faculty member). Dur and Uṅver [2] show that balancedness may lead to impossibility results, when associated with different natural axioms, like individual rationality or fairness. Their analysis focuses on efficiency and they propose a new procedure based on the Top Trading Cycle algorithm (rather than the Deferred Acceptance algorithm). Biró, Klijn and Papai [21] also focus attention on an extension of the TTC algorithm to analyze student exchange programs where a balancedness condition holds. In a recent paper, Hafalir, Kojima and Yenmez [23] study an interdistrict school choice problem, considering three sides of the market: districts, students and schools. Their model, however, is essentially a two-sided model since they do not model the priorities of schools.

Other nested algorithms. Concerning refugee resettlement, Delacrétaz, Kominers, and Teytelboym [24] generalize matching models with distributional constraints by considering multidimensional attributes which capture the services (housing, school seats, ...) that localities can provide to refugee families. To generate (quasi) stable allocations, they propose the Maximum Rank Deferred Acceptance (MRDA) algorithm; this algorithm includes a nested phase under the families' rank is updated instead of assigning objects among localities, as it happens in the NDA's nested phase.

Regional preferences in doctor-hospital matching. In the paper most closely related to ours, Kamada and Kojima [4] analyze the assignment of doctors to hospitals in Japan, allowing for some flexibility across hospitals in the same region. They also consider a three-sided matching problem, where regions have preferences over the assignment of doctors to hospitals within the region. There are two main differences between Kamada and Kojima [4] and our model of matching through institutions. First, hospitals have priorities over doctors in Kamada and Kojima [4] whereas objects have priorities over institutions in our model. Second, and more importantly, regional preferences in Kamada and Kojima [4] are defined over the distribution of doctors over hospitals as measured by a vector of capacities whereas we consider that institutions care about the precise assignment of households to objects. Kamada and Kojima [4] propose a Generalized Flexible Deferred Acceptance (GFDA) algorithm to match doctors to hospitals. In an Appendix, we show that the NDAI algorithm can be adapted to their model in order to produce fair assignments.

Social housing. Finally, we note that the assignment of social housing that motivated our study has recently been analyzed in a series of papers (Leshno [25], Bloch and Cantala [26], Schummer [27] and Thakral [28]) which focus on very different aspects of the problem the revelation of persistent information on types in Leshno [25], the dynamic sequence of decisions in Bloch and Cantala [26], the manipulation of orders in Schummer [27] and multiple waiting list mechanisms in Thakral [28].

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents preliminary results on the existence of fair assignments and introduces the over-demand condition. Section 4 describes the Nested DA algorithm, and Section 5 contains our main results. Section 6 discusses two extensions of the model and the relation with Kamada and Kojima [4]'s model of regional preferences. Section 7 contains our concluding comments. The appendix contains details of the examples, complete proofs of the results, and details of the use of the NDAI in the model of regional preferences of doctors and hospital assignments in Japan.

2. The Model

A matching market with institutions is a three sided-market where

1. $I = \{i_1, i_2, .., i_N\}$ is a finite set of institutions, a generic institution is i;
2. $A = \{a_1, ..., a_A\}$ is a finite set of apartments, a generic apartment is a;
3. $H = \{h_1, ..., h_H\}$ is a finite set of households, a generic household is h.

Each apartment is attached to an institution through the mapping $\alpha : A \to I$. We let A_i denote the set of apartments owned by institution i with cardinality m_i. Similarly, each household is attached to an institution through the mapping $\tau : H \to I$. Again, we let H_i denote the set of households belonging to institution i with cardinality n_i. We assume throughout, as is the case in most applications of the model, that the number of households attached to institution i is at least as large as the number of apartments owned by institution i: $n_i \geq m_i$.

Households have preferences over the set of apartments, A, denoted by R_h, with strict preferences P_h. We let $\mathbf{R}$ denote the profile of households' preferences. We say that an apartment a is acceptable for h if $aP_h\varnothing$.

Institutions have strict preferences over the matchings between apartments and households belonging to the institution. We let $\succ_i$ denote the preferences of institution i over sets of pairs (a, h) such that $h \in H_i$. We assume that for all $i \in I$ the preference $\succ_i$ is responsive, i.e. for all subsets of pairs $U \in 2^{A\times H_i}$ and all pairs (a_r, h_r), $(a_s, h_s) \in (A \times H_i) \setminus U$ we have that

i. $U \cup \{(a_r, h_r)\} \succ_i U \cup \{(a_s, h_s)\}$ if and only if $\{(a_r, h_r)\} \succ_i \{(a_s, h_s)\}$, and

ii. $U \cup \{(a_r, h_r)\} \succ_i U$ if and only if $\{(a_r, h_r)\} \succ_i \varnothing$.

We write $(a, h) \succ_i \varnothing$, instead of $\{(a, h)\} \succ_i \varnothing$, when (a, h) is acceptable for institution i. The preferences of institutions over matchings take into account different elements: (i) the preferences of households belonging to the institution, (ii) prioritization criteria needed to select among different households applying for the same apartment and (iii) objective measures – such as the size of the household and of the apartment- - which do not necessarily enter household preferences.

Apartments have priorities over the institutions, denoted by π_a. If apartment a is owned by institution i, then institution i is at the top of the priority. We let $\pi = (\pi_a)_{a\in A}$ denote the profile of priorities of apartments over institutions.

A matching market with institutions is thus defined by an 8-tuple $E =< I, A, H, \alpha, \tau, \mathbf{R}, \succ, \pi >$.

When matching markets are intermediated by institutions, matchings are not simply defined as two-sided matchings between households and apartments, but as three-sided matchings associating households, institutions and apartments. The mechanism we propose consists in two separate assignments: we first assign apartments to institutions, then assign households to apartments. We thus describe an assignment as: (i) a many-to-one matching between apartments and institutions, and (ii) a one-to-one matching between households and apartments. These assignments are formalized in the following definition.

An **assignment** is a pair $\mu = (\theta, \varphi)$ such that:

i. $\theta : A \cup I \to 2^A \cup I \cup \{\varnothing\}$ where

  i.a $\theta(a) \in I \cup \{\varnothing\}$,

  i.b $\theta(i) \in 2^A$,

  i.c $a \in \theta(i)$ if and only if $\theta(a) = i$;

ii. $\varphi : A \cup H \to A \cup H \cup \{\varnothing\}$, where

  ii.a $\varphi(a) \in H \cup \{\varnothing\}$,

  ii.b $\varphi(h) \in A \cup \{\varnothing\}$,

  ii.c $\varphi(h) = (a)$ if and only if $\varphi(a) = h$.

iii. $\theta(a) = \tau(\varphi(a))$ for all $a \in A$.

If $\mu = (\theta, \varphi)$ is an assignment, we denote by $\varphi(h)$ the apartment assigned to household h and by $\mu(i) = \{(a,h) | a \in \theta(i), h \in H_i\}$ the matchings of households in H_i to apartments in A.

Conditions i. a, b and c define the many-to-one matching θ between apartments and institutions, reflecting the way in which institutions reassign apartments among themselves. Conditions ii. a, b and c define the one-to-one matching φ between households and apartments. Condition iii. is a consistency condition between the two matchings. It requires that if an household is matched to an apartment a in φ, the institution to which it belongs is matched to the same apartment a in θ.

We next define the choice rules of institutions. Consider $\mathcal{U}_i$, the collection of any subset of matchings U with the property that any household and apartment appear at most once in U, i.e. $\mathcal{U}_i = \{U \in 2^{A \times H_i} | (a,h) \in U \Rightarrow (a',h) \notin U, (a,h') \notin U \text{ for } a' \neq a, h' \neq h\}$. For any institution i, the set $\mathcal{U}_i$ collects all one-to-one matchings between any subset of agents in H_i and apartments in A. We define the **choice rule** Ch_i of institution i, given the number of apartments m_i, as a mapping choosing the pairs with the highest priority for i in $\mathcal{U}_i$: for all $(U, m_i) \in 2^{A \times H_i} \times \mathbb{Z}_+$, the choice of i is the set $Ch_i(U, m_i) = \max_{\succ_i} \{u \subseteq U \mid |u| \leq m_i \text{ and } u \in \mathcal{U}_i\}$. The choice rule of institution i thus selects, given a set of possible matches U and the quota m_i, the best matches available in the set U.

We now extend classical properties of the assignment μ to matching with institutions. An assignment μ is **individually rational** if

i. for all $h \in H$ $\varphi(h) R_h \varnothing$

ii. $\mu(i) = Ch_i(\mu(i), m_i)$.[5]

In words, an assignment is individually rational if it satisfies two conditions. First, all households weakly prefer their assignment under μ to remaining unmatched. Second, the assignment of each institution either is a set of acceptable household-apartments pairs whose cardinality does not exceed the institution' quota, or the empty set. In an individually rational matching, the two sides of the market which are endowed with preferences– households and institutions – prefer the outcome of the matching μ to what they would obtain by not participating in the matching.

We now turn to the distributional constraints faced by institutions. An individually rational assignment $\mu = (\theta, \varphi)$ is **feasible if** $|\theta(i)| = m_i$ for all $i \in I$. In a feasible assignment, each institution receives the same number of apartments as the number of apartments they initially own.

An assignment μ is **non-wasteful** if no household-institution pair (h, i) can claim an empty apartment a, i.e. there is no i, h and a such that:

i. $a P_h \varphi(h)$,

ii. $(a,h) \in Ch_i(\mu(i) \cup \{(a,h)\}, m_i)$, and

[5]Since priorities are responsive, this means that, for all $i \in I$, either $(a,h) \succ_i \varnothing$ for all $(a,h) \in \mu(i)$, or $\mu(i) = \varnothing$.

iii. $\theta(a) = \varnothing$.

A household-institution pair (h, i) has **justified envy** over the household-institution pair (h', i') at the individually rational assignment μ if $i = \tau(h)$, $i' = \tau(h')$, and $\varphi(h') = a \in \theta(i')$, such that

i. $aP_h\varphi(h)$,

ii. $(a, h) \in Ch_i(\mu(i) \cup \{(a, h)\}, m_i)$, and

iii. $i\pi_a i'$ or $i = i'$.

We thus define justified envy from *pairs* of household and institution. Consider first two households attached to the same institution, the pair (h, i) has justified envy over the pair (h', i) if and only if (i) household h prefers the apartment assigned to h' to its current match and (ii) institution i prefers to assign apartment a to household i. If households h and h' belong to different institutions, the pair (h, i) has justified envy over the pair (h', i') if (i) household h prefers the apartment assigned to h' to its current match, (ii) institution i prefers to match h to the apartment of h' rather than to its current match and (iii) institution i has priority over institution i' for the apartment currently occupied by h'.

Notice that, following the deviation, the assignment will not satisfy the distributional constraints if the apartment a is assigned to a different institution. If we only consider deviations which do not induce a change in the assignment of apartments to institutions, we obtain a weaker form of envy, called justified envy over households of the same institution:

There is **justified envy over households of the same institution** when a pair (h, i) has justified envy over a pair (h', i) at an individually rational assignment μ. This concept only considers justified envy among households attached to the same institution.

We now define fair and efficient assignments. An assignment μ is **fair** if it is individually rational, non-wasteful and there is no justified envy. A matching μ is **fair over households of the same institution** if it is individually rational, non-wasteful and there is no justified envy for households of the same institution. Notice that the quota of institution i, m_i, appears in the definitions of individual rationality, non-wastefulness and justified envy. Hence fairness takes into account the quotas faced by each institution. As is usual in the literature on matching with distributional constraints ([**?**]), we distinguish between feasible matchings (which satisfy the quotas) and fair matchings (where quotas are implicitly taken into account). We define Pareto efficiency by restricting attention to preferences of households, and ignoring the preferences of institutions. An assignment μ is **Pareto efficient** if there is no matching μ' such that all households weakly prefer μ' to μ, with strict preference for at least one household. An assignment μ' **Pareto dominates** another assignment μ if $\mu'(h)R_h\mu(h)$ for each $h \in H$, and $\mu'(h')P_{h'}\mu(h')$ for at least one $h' \in H$.

Finally we consider the incentives of households to reveal their true preferences: A **mechanism** Λ associates an assignment μ to every profile of preference $\mathbf{R}$. Let R_{-h} denote the profile of preferences of all households except household h.

A mechanism Λ is **strategy-proof for households** if truth-telling is a dominant strategy for all households, i.e. $\Lambda[R_h, R_{-h}](h) R_h \Lambda[R'_h, R_{-h}](h)$ for all h, for all $R'_h \in \mathcal{R}_h$ and for all $R_{-h} \in \mathcal{R}_{-h}$.

3. Existence and fairness

In this Section, we study existence and fairness of feasible assignments in the three-sided matching market. Even under the Assumption **??**, as is common in the literature on distributional constraints, feasible assignments may not exist. Moreover, if feasible assignments exist, distributional constraints may lead to inconsistencies in terms of fairness.

3.1. Existence of feasible assignments

The following simple example shows that distributional constraints may conflict with the preferences of institutions.

Example 3.1. Let $I = \{i_1\}$, $A = \{a_1, a_2\}$, $H = \{h_1, h_2\}$. The profiles of institutions' priorities $\succ$ and households' preferences P are given by

$\succ_{i_1}$
(a_1, h_1)
(a_1, h_2)

P_{h_1}	P_{h_2}
a_1	a_1
a_2	a_2

In this simple example, apartment a_2 is not acceptable by the institution. This implies that the exact distributional constraints cannot be satisfied in an individually rational assignment. Hence there is no feasible assignment in this example. □

The next example shows that feasible assignments may not exist even if all apartments are acceptable for at least one institution, so the conflict between distributional constraints and preferences of institutions is more complex than in the previous example.

Example 3.2. Let $I = \{i_1, i_2\}$, $\{a_1, a_2, a_3\}$, $H = \{h_1, h_2, h_3, h_4, h_5\}$. Let $A_1 = \{a_1, a_2\}, A_3 = \{a_3\}$, $H_1 = \{h_1, h_2, h_3\}$ and $H_2 = \{h_4, h_5\}$. The profiles of institutions' priorities $\succ$ and households' preferences P are given by

$\succ_{i_1}$	$\succ_{i_2}$
(a_1, h_1)	(a_1, h_4)
(a_2, h_1)	(a_3, h_5)
(a_3, h_1)	

P_{h_1}	P_{h_2}	P_{h_3}	P_{h_4}	P_{h_5}
a_1	a_1	a_1	a_1	a_1
a_2	a_3	a_3	a_2	a_2
a_3	a_2	a_2	a_3	a_3

In this example, the difficulty does not stem from the fact that institutions judge the apartments unacceptable, but from the fact that institution 1 only cares about household h_1. Hence, in an individually rational assignment, h_1 is the only household matched in H_1 and the distributional constraint, $|\{h \in H_1 | \varphi(h) \neq \varnothing\}| = 2$ cannot be satisfied. □

3.2. Fairness

Even when feasible assignments exist, they may not satisfy fairness due to the preferences of institutions. We illustrate this point in the following example.

Example 3.3. Consider $I = \{i_1, i_2\}$, $H = \{h_1, h_2, h_3\}$ and $A = \{a_1, a_2, a_3\}$. Let $A_1 = \{a_1, a_2\}$, $A_2 = \{a_3\}$, $H_1 = \{h_1, h_2\}$ and $H_2 = \{h_3\}$. The profiles of institutions' priorities, households' preference and apartments' priorities are

$\succ_{i_1}$	$\succ_{i_2}$
(a_2, h_2)	(a_1, h_3)
(a_1, h_1)	
(a_1, h_2)	
(a_3, h_1)	

P_{h_1}	P_{h_2}	P_{h_3}
a_1	a_1	a_1
a_2	a_2	a_3
a_3	a_3	a_2

π_{a_1}	π_{a_2}	π_{a_3}
i_1	i_1	i_2
i_2	i_2	i_1

Notice that (a_1, h_3) is the unique acceptable assignment of institution i_2 and (a_2, h_1) is not acceptable for institution i_1. Therefore, in this market there exists one and only one feasible assignment which is:

$$\mu = \begin{pmatrix} h_1 & h_2 & h_3 \\ a_3 & a_2 & a_1 \\ i_1 & i_1 & i_2 \end{pmatrix}.$$

We now claim that the only feasible assignment μ is not fair. Notice that i) $a_1 P_{h_1} a_3$, ii) $(a_1, h_1) \in Ch_1(\mu(1) \cup \{(a_1, h_1)\}, m_1)$ because $(a_1, h_1) \succ_{i_1} (a_3, h_1)$, and iii) $1 \pi_{a_1} 2$. Hence, the pair (h_1, i_1) has justified envy over the pair (h_3, i_2) at the apartment a_1. $\square$

The previous examples build on the fact that distributional constraints may conflict with the preferences of households and institutions. In Example 3.3, the unique feasible assignment is upset by a deviation which results in one of the two institutions not fulfilling its quota.

Assuming that every apartment is acceptable to any household and institution would clearly overcome the difficulties highlighted by Examples 3.1, 3.2 and 3.3. It is possible to provide a weaker requirement which guarantees the existence of a fair assignment satisfying distributional constraints. It suffices to assume that, for each institution and each apartment, there exists a household who is willing to accept the apartment, and such that the institution also accepts the assignment. This leads us to define the following over-demand condition, under which fair assignments satisfying the distributional constraints will be shown to exist.

Assumption 1. (**Over-demand Condition**) For all individually rational assignments $\mu = (\theta, \varphi)$, for all institutions i and apartments a, there exists a household h such that $\varphi(h) = \varnothing$ (h is unassigned under μ), $(a, h) \succ_i \varnothing$, $a P_h \varnothing$ and $i \pi_a \varnothing$.

The over-demand condition is not defined in terms of the primitives of the model, but is easy to interpret using the definition of a matching market. It is satisfied whenever there is a large imbalance between the number of households and apartments, and for each apartment a, one can find an institution and a household who is unmatched under the individually rational matching μ such that both the institution and the household find apartment a acceptable. This condition holds when all apartments are acceptable to some household, and when there is an overload of household in the matching market. We will later prove (by construction using the NDA algorithm) that the over-demand condition is a sufficient condition for the existence of a feasible and fair assignment in a matching through institutions when distributional constraints are active.

3.3. No lattice structure

Finally, we note that the lattice structure of fair matchings, which prevails in two-sided markets, may fail in matching with institutions. Households may disagree on the best fair assignment satisfying distributional constraints.

Example 3.4. Consider a market such that $H = \{h_1, h_2\}$, $A = \{a_1, a_2\}$ and $I = \{i_1, i_2\}$, $A_1 = \{a_2\}, A_2 = \{a_1\}$, $H_1 = \{h_1\}, H_1 = \{h_2\}$. Institutions priorities $\succ$, households' preferences P and apartments' priorities π are given by.

$\succ_{i_1}$	$\succ_{i_2}$	P_{h_1}	P_{h_2}	π_{a_1}	π_{a_2}
(a_1, h_1)	(a_2, h_2)	a_1	a_1	i_2	i_1
(a_2, h_1)	(a_1, h_2)	a_2	a_2	i_1	i_2

In this market, there are two fair assignments satisfying the distributional constraints:

$$\mu = \begin{pmatrix} h_1 & h_2 \\ a_1 & a_2 \\ i_1 & i_2 \end{pmatrix} \text{ and } \mu' = \begin{pmatrix} h_2 & h_1 \\ a_1 & a_2 \\ i_2 & i_1 \end{pmatrix}.$$

Household h_1 prefers μ' to μ, whereas household h_2 has the reverse preferences. Therefore, the set of fair assignments does not possess a lattice structure.

4. The Nested Deferred Acceptance Mechanism

In this Section, we extend the deferred acceptance algorithm of Gale and Shapley [29] to generate a fair assignment between households, institutions, and apartments. We introduce the Nested Deferred Acceptance algorithm (NDA), which produces an assignment $\mu = (\theta, \varphi)$ in the matching market through institutions.

The idea behind the NDA is to compute simultaneously a many-to-one matching, θ, assigning apartments to institutions, and a one-to-one matching, φ, assigning apartments to households by nesting two deferred acceptance algorithms. In the main DA iteration (the "outer loop "), each unassigned household applies to her most preferred apartment. Given these demands, we run another DA (the "inner loop") where each institution demands a set of apartments. The procedure continues iteratively until all apartments are assigned to households. In the text, we present a general overview of the NDA and relegate the full details of this algorithm in Appendix A.

Initialization

Consider a market $E =< I, A, H, \alpha, \tau, \mathbf{R}, \succ, \pi >$. The assignment is initialized to be the empty assignment, i.e., each household, institution, and apartment is unassigned at the beginning of the assignment procedure.

$\mathbf{A}^t$. The outer loop

At step t, household h applies to her most preferred acceptable apartment among those which have not yet rejected her. If a household is matched to her preferred apartment, she just reiterates her application. Let $D^t(h)$ denote the application of household h at that stage.

Each institution i observes applications by the households in H_i. It constructs the set $\mathcal{M}_i^t$ of all possible matchings (a, h) where $a = D^t(h)$ for all $h \in H_i$. The inner loop then starts.

$\mathbf{B}^t$. The inner loop

In this phase, institutions compete to form household-apartment pairs that are maximal according to their preferences in the choice set $\mathcal{M}_i^t$. We initialize the inner loop by assuming that no apartment has been assigned to any institution yet, and each institution has preferences over the entire set $\mathcal{M}_i^t$, $\mathcal{M}_i^{t,1} = \mathcal{M}_i^t$.

At stage s of the inner loop, each institution applies to the set of apartments which are maximal in the set of possible matches $\mathcal{M}_i^{t,s}$ under the quota m_i. (step $\mathbf{B}^{t,s}$**.1**). In the next step (step $\mathbf{B}^{t,s}$**.2**), apartments receive applications from institutions. Each apartment retains the application from the institution with the highest priority according to π_a.

At the end of step $\mathrm{B}^{t,s}.2$, if all institutions fill their quotas, the inner loop stops. If not, some apartments must be unassigned and the DA among institutions is repeated. In the next iteration of the inner loop, if institution i was rejected from apartment a, all the household-apartment pairs (a, h) are deleted from the set of possible matches of institution i. So $\mathcal{M}_i^{t,s+1} = \mathcal{M}_i^{t,s} \setminus \{(a, h)|$ apartment a was not assigned to institution i at stage $B^{t,s}\}$.

The inner loop stops when each institution has fulfilled its quota, or the remaining apartments are unacceptable (step $\mathbf{B}^t$**.3**). [6]

$\mathbf{C}^t$. Ending conditions

The algorithm then returns to the outer loop. In the tentative assignment reached at step $\mathbf{B}^t$**.3**, if a household has been rejected from the apartment at stage $\mathbf{A}^t$, she updates the set of apartments which have not yet rejected her, and the algorithm moves to step $\mathbf{A}^{t+1}$. If every

[6] The NDA algorithm thus takes into account the distributional constraints both in the computation of the list of apartments sent by institutions and in the stopping rule of the inner loop. If all apartments are acceptable, the NDA requires that all institutions exactly fill their quotas at the end of the inner loop.

household is assigned to some apartment, or has been rejected by all her acceptable apartments, the NDA stops.

We denote by $\mu^{NDA}[E] = (\theta^{NDA}[E], \varphi^{NDA}[E])$ the output of the Nested Deferred Acceptance algorithm at the matching market E. We use $NDA[P]$ to denote the output of the NDA algorithm under the preference profile P. Notice that the NDA algorithm has a finite number of steps because each DA ends in finite time.

4.1. An example of NDA

The following example shows how the NDA algorithm works.

Example 4.1. Consider the market $I = \{i_1, i_2\}$, $H = \{h_1, h_2, h_3\}$ and $A = \{a_1, a_2, a_3\}$. Institution 1 owns apartment a_1 and institution 2 owns apartment a_2; so, $A_1 = \{a_1\}$ and $A_2 = \{a_2\}$. Household h_1 is attached to institution 1 whereas households h_2 and h_3 are attached to institution 2, $H_1 = \{h_1\}, H_2 = \{h_2, h_3\}$. The profiles of institutions' preferences $\succ$, households' preferences P, and apartments' priorities π are given by

$\succ_{i_1}$	$\succ_{i_2}$
(a_1, h_1)	(a_1, h_2)
(a_2, h_1)	(a_2, h_3)
	(a_2, h_2)

P_{h_1}	P_{h_2}	P_{h_3}
a_1	a_1	a_2
a_2	a_2	a_1

π_{a_1}	π_{a_2}
i_1	i_2
i_2	i_1

At step A^1, households announce their preferred apartment: h_1 and h_2 announce a_1, and h_3 announces a_2. The NDA then moves to step B^1.1 with $\mathcal{M}^{1,1}_{i_1} = \{(a_1, h_1)\}$ and $\mathcal{M}^{1,1}_{i_2} = \{(a_1, h_2), (a_2, h_3)\}$. During the first stage of the inner loop, each institution chooses the set of apartment-household pairs taking its quota into account: i_1 announces $\{(a_1, h_1)\}$ and i_2 announces $\{(a_1, h_2)\}$. Since a_1 is demanded by both institutions, the algorithm moves to step B^1.2. As institution i_1 has priority over i_2 for apartment a_1, a_1 rejects i_2; so, $\mathcal{M}^{1,2}_{i_1} = \{(a_1, h_1)\}$ and $\mathcal{M}^{1,2}_{i_2} = \{(a_2, h_3)\}$. Thus, i_1 announces $\{(a_1, h_1)\}$ and i_2 announces $\{(a_2, h_3)\}$ during B^{2,2}.2. Since institutions ask for different apartments, they fill their quotas and the algorithm goes to C^1. The step 1 of the NDA outputs the following assignment

$$\mu^1 = \begin{pmatrix} h_1 & h_2 & h_3 \\ a_1 & \varnothing & a_2 \\ i_1 & \varnothing & i_2 \end{pmatrix}.$$

The NDA moves back to the outer loop and household h_2 announces apartment a_2 at phase A^2. The algorithm moves now to B^2.1 where institution i_2 announces that it prefers (a_2, h_3) to (a_2, h_2), and the pairs (a_1, h_1), (a_2, h_3) are matched. This is the last step of the algorithm as household h_2 has exhausted the list of apartments, and the NDA's final output is $\mu^{NDA} = \mu^1$.

Remark 1. Example 4.1 illustrates the necessity of the inner loop. Consider a counterfactual assignment rule where institutions are only asked to report their preferences once. At the end of step A^1, we compute $\mathcal{M}^1_{i_1} = \{(a_1, h_1)\}$ and $\mathcal{M}^1_{i_2} = \{(a_1, h_2), (a_2, h_3)\}$. Since both institutions are constrained by their quotas and can only request one apartment, we have that $Ch^1_{i_1}(\mathcal{M}^1_{i_1}, 1) = \{(a_1, h_1)\}$ and $Ch^1_{i_2}(\mathcal{M}^1_{i_2}, 1) = \{(a_1, h_2)\}$. If there is no inner loop, household h_3 is then rejected from apartment a_2, institution i_1 has priority over apartment a_1 and assigns a_1 to h_1, and apartment a_2 is not assigned. The algorithm moves to step A^2. At that step, households h_2 and h_3 ask for a_2 and a_1 respectively. Since (a_1, h_3) is not acceptable for institution i_2, the algorithm finally outputs the assignment (a_1, h_1) and (a_2, h_2). However, note that (h_3, i_2) has justified envy over (h_2, i_2) at apartment a_2.

4.2. Participation Criterion

We first show that institutions have an incentive to pool the apartments in order to maximize the preferences of the households. We consider the possibility that institution i reserves some apartments for the exclusive usage of its households. The set of all apartments owned by i is given by A_i; if i **does not pool** an apartment $a \in A_i$, this implies that $\varnothing \pi_a j$ for all $j \neq i$. If, on the other hand, i pools the apartment a, there exists $j \neq i$ such that $j \pi_a \varnothing$. We denote by $\tilde{\pi}$ the profile of apartments priorities where no institution pools any of its apartments.

We say that the assignment procedure satisfies the **participation criterion** when pooling apartments generates an assignment $\mu = (\theta, \varphi)$ that is weakly preferred by all households to the autarky assignment, i.e., $\varphi_A[\pi](h) R_h \varphi_A[\tilde{\pi}](h)$ for all $h \in H$.

Proposition 4.1. *The NDA assignment procedure satisfies the participation criterion.*

Proposition 4.1 shows that, when the NDA algorithm is run, households cannot be harmed by pooling objects. Pooling objects increases the list of households which are acceptable for any apartment, resulting in weakly better assignments for households at the end of the NDA algorithm.

4.3. No distributional constraints

In this subsection we consider the hypothetical benchmark where there are no distributional constraints, no balance between the number of households belonging to institution i who receive an apartment and the number of apartments owned by institution i. We first show that the "inner loop" (phase B^t) of the NDA algorithm only requires one step.

Lemma 1. *Consider a matching market with institutions and no distributional constraints. Then, phase B^t of the NDA algorithm is iterated only once.*

In the absence of distributional constraints, if an apartment is tentatively assigned to a household and institution at some step during the algorithm, it is assigned to some household-institution pair in the final assignment, not necessarily the same household-institution pair.

Lemma 2. *Consider a matching market with institutions and no distributional constraints. If an apartment is assigned at some step t by some institution, this apartment is assigned under the assignment μ^{NDA}.*

The following theorem shows all the desirable properties that the NDA algorithm satisfies when there are no distributional constraints.

Theorem 4.1. *Consider a matching market with institutions* $< I, A, H, \alpha, \tau, \mathbf{R}, \succ, \pi >$ *with no distributional constraints.*

1. *The* μ^{NDA} *assignment is individually rational, non-wasteful and there is no justified envy; namely, the assignment* μ^{NDA} *is fair.*
2. *There is no fair assignment that Pareto dominates* μ^{NDA}.
3. *The NDA mechanism is strategy-proof for households.*

Theorem 4.1 shows that in the absence of distributional constraints, the NDA in the model of matching through institutions inherits the properties of DA in classical school choice problems. The exchange of apartments across institutions can be done freely, so that the problem resembles a classical two-sided matching market problem, where households have preferences over apartments, and priorities over apartments are determined by the preferences of institutions.

5. Dealing with distributional constraints

When distributional constraints are present, the NDA algorithm takes into account the quotas to assign apartments to institutions during the inner loop. As opposed to the case studied in the previous Subsection, institutions tentatively accept apartments to fill their quotas at the end of the inner loop of the algorithm. This may prevent the NDA from producing feasible and fair assignments. During Phase B^t of the NDA algorithm, assuming that all apartments are acceptable, institutions fill their quotas exactly. Later in the run of the NDA, better options might arise for some institutions, leading them to drop some apartments which were demanded by other institutions in earlier rounds of the algorithm. In this case, institutions act as interrupters, temporarily accepting household-apartments pairs that will be dropped in the final outcome. As in school choice problems, this will temporarily bar other institutions from accessing apartments and may prevent the emergence of fair outcomes.

5.1. Interrupters

The following example shows that the NDA may produce an assignment which does not satisfy fairness. We show that such problem arises given the existence of institutions that made *interruptions*, which generates justified envy over households that belong to the same institution.

Example 5.1. (There is justified envy over households that belong to the same institution). Let $I = \{i_1, i_2\}$, $A = \{a_1, a_2, a_3, a_4\}$, $A_1 = \{a_3, a_4\}$, $A_2 = \{a_1, a_2\}$, and $H = \{h_1, h_2, \ldots, h_7\}$. Also, we consider τ such that $H_1 = \{h_1, h_2, h_5, h_6\}$ and $H_2 = \{h_3, h_4, h_7\}$. The profiles of institutions' priorities $\succ$ and households' preferences P are

$\succ_{i_1}$	$\succ_{i_2}$
(a_1, h_1)	(a_2, h_3)
(a_1, h_2)	(a_2, h_4)
(a_2, h_1)	(a_1, h_3)
(a_2, h_2)	(a_1, h_4)
(a_1, h_5)	(a_1, h_7)
(a_2, h_6)	(a_2, h_7)
(a_3, h_1)	
(a_3, h_6)	
(a_4, h_6)	

P_{h_1}	P_{h_2}	P_{h_3}	P_{h_4}	P_{h_5}	P_{h_6}	P_{h_7}
a_1	a_2	a_1	a_1	a_2	a_1	a_1
a_2	a_1	a_2	a_2	a_1	a_2	a_2
a_3	a_3	a_4	a_3	a_3	a_4	a_3
a_4	a_4	a_3	a_4	a_4	a_3	a_4

Also, we consider the following profile of apartments' priorities π

π_{a_1}	π_{a_2}	π_{a_3}	π_{a_4}
i_2	i_2	i_1	i_1
i_1	i_1	i_2	i_2

Running the NDA algorithm (see Appendix B for the detailed steps), we obtain the following assignment

$$\mu^{NDA} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_3 & a_1 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ i_1 & i_1 & i_2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

Notice that the sets $\{(a_1, h_5), (a_2, h_6), (a_3, h_3)\}$ and $\{(a_1, h_4), (a_2, h_4), (a_1, h_7), (a_2, h_7)\}$ are sets of unassigned pairs that are acceptable for institutions i_1 and i_2, respectively. Hence the over-demand condition holds. Moreover, note that $a_1 P_{h_1} a_3$ and $\{(a_1, h_1)\} \succ_{i_1} \{(a_1, h_2)\}$, where $\varphi_I^{NDA}(h_1) = \varphi_I^{NDA}(h_2) = 1$, $\varphi_A^{NDA}(h_2) = a_1$ and $\varphi_A^{NDA}(h_1) = a_3$. Thus, the pair (h_1, i_1) has justified envy over (h_2, i_1) at apartment a_1. The assignment μ^{NDA} is thus not fair. $\square$

Example 5.1 describes a market where the NDA fails to generate a fair assignment, even when the over-demand condition holds. Notice that the assignment is not fair because there is justified envy over households of the same institution. The problem arises because of an interruption made by one institution. During the run of the NDA, institution i_2 is tentatively assigned to a_1 at step 1. By this tentative assignment, household h_1 and institution i_1 are displaced from a_1 at the end of step 1. Similarly, household h_3 and institution i_2 are displaced from apartment a_1 at step 2 (see Table B.4 in Appendix B). Later, at step 3, institution i_1 asks for a_1 and obtains it because $i_1 \pi_{a_1} i_2$. However, notice that h_1 can no longer demand this apartment because she was rejected from it at step 1. As a consequence, the choice function of institution i_1 does not consider the pair (a_1, h_1), which results in justified envy between the pairs (h_1, i_1) and (h_2, i_1).

Summarizing the previous observation, we see that apartment a_1 is tentatively assigned to institution i_2 during steps 1 and 2 of the NDA, but not in the final assignment. In the terminology of Kesten [3], institution i_2 is an interrupter for apartment a_1. Formally,

Definition 1. Given a matching problem to which the NDA is applied, we say that an institution i is an **interrupter** for apartment a if there exists

1. steps t to $t+n$ such that $a \in \theta^{t'}(i)$ for all $t' \in \{t, t+1, \ldots, t+n\}$ but $a \notin \theta^{t'}(i)$ for all $t' > t+n$, and
2. an institution $j \neq i$ and a household h such that $(a,h) \in Ch_j(\mathcal{M}_j^{t'}, m_j)$ but $(a,h) \notin \mu^{t'}(j)$ for some $t' \in \{t, t+1, \ldots, t+n\}$.

Although the definition of interruption is the same as in Kesten [3], the role that interrupters play is different. In Kesten [3], students are the agents who request seats, are responsible for the interruption and cause a loss of efficiency in the DA. In our case, institutions are the ones who make interruptions resulting in a failure of fairness. The interruption is due to the fact that institutions are intermediaries, acting as receivers of the demand of households, on which they construct their own demand on apartments. In the following section, we show that the deletion of interrupters in our setting restores fairness.

5.2. Nested Deferred Acceptance with Interrupters

To overcome the problem caused by interruptions, we modify the NDA by introducing a second stage where we search for all interrupter institutions, as in [3]. We then delete from the preferences of these institutions the pairs containing the apartment where they cause the interruption. Notice that Kesten [3] defines this operation over students' preferences (the equivalent in our model of households in the outer loop), because he identifies students as the source of inefficiency in the Deferred Acceptance algorithm. In our case, we delete apartments from the preferences of institutions, which are involved in the inner loop of the algorithm, as they are the source of justified envy in the NDA algorithm.

Let $\Im$ be the set of all possible priorities $\succ_i$, for all $i \in I$. The **delete operation** over $\Im$ is the function $\setminus : \Im \times (A \times H) \to \Im$ such that $\setminus(\succ, a)$, or simply $\succ \setminus a$, is the priority that declares all pairs $(a,h) \succ_i \varnothing$ as unacceptable for i. In other words, the priority $\succ \setminus a$ drops all acceptable pairs (a,h) from $\succ_i$ and holds the original order in the priority $\succ_i$.

We now formally define the Nested Deferred Acceptance with Interrupters (NDAI). Each step of this mechanism has two stages: the NDA algorithm runs in the first stage, while the second stage deletes pairs from the priorities of interrupters. The NDAI proceeds as follows.

Initialization Initialize the counter of iterations over interrupter institutions at $x := 0$.

Step 0. This step is divided in the following stages:

Stage 0.1 NDA Phase. Let $\succ^0 = (\succ^{i,0})_{i\in I} = (\succ_i)_{i\in I}$. Run the NDA algorithm using the profile of priorities and preferences $(\succ^0, P)$.

Stage 0.2 Deletion in Priorities If there are no interrupters at Stage 0.1, the algorithm stops. Otherwise, find the last step of the NDA phase at which an interrupter is rejected from the apartment for which it is an interrupter. For each interrupter institution i, $\succ^{i,1} = \succ^{i,0} \setminus a$; $\succ^{j,1} = \succ^{j,0}$ if j is not an interrupter.

Step x. This step is divided in the following stages:

Stage x.1 NDA Phase. Run the NDA algorithm with the profile of priorities and preferences $(\succ^x, P)$.

Stage x.2 Deletion in Priorities. If there are no interrupters at Stage x.1, the algorithm stops. Otherwise, find the last step of the NDA phase at which an interrupter is rejected from the apartment for which it is an interrupter. For each interrupter institution i, $\succ^{ix+1} = \succ^{i,x} \setminus a$; $\succ^{j,(x+1)} = \succ^{j,x}$ if j is not an interrupter.

The output of the previous mechanism is denoted by μ^{NDAI}. The NDAI is solvable in finite time because each NDA phase is solvable in a finite number of steps, and there are at most $|I|$ interrupters. To illustrate how the NDAI algorithm works, we continue with the analysis of Example 5.1.

Example 5.2. We consider the same market as in the Example 5.1. We recall that institution i_2 causes an interruption over the pair (a_1, h_1). We delete all acceptable pairs that include a_1 from the priority $\succ^2$. (Appendix B details the complete run of the NDAI algorithm.)

At stage 1.1 there is no interrupter, and the algorithm finishes. The NDAI mechanism outputs the following assignment

$$\mu^{NDAI} = \mu^4 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & a_3 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ i_1 & i_1 & i_2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

Notice that μ^{NDAI} is indeed a fair assignment. □

The next Theorem shows that the assignment produced by the NDAI algorithm satisfies a number of desirable properties.

Theorem 5.1. *Consider a matching market with institutions $E =< I, A, H, \alpha, \tau, \mathbf{R}, \succ, \pi >$ that satisfies the over-demand condition.*

1. *The μ^{NDAI} is individually rational, non-wasteful, respects distributional constraints and there is no justified envy; namely, the assignment μ^{NDAI} is feasible and fair.*
2. *There is no fair assignment which Pareto dominates μ^{NDAI}.*
3. *The NDAI is strategy-proof for households and satisfies the participation criterion.*

Theorem 5.1 extends the classical properties of the deferred acceptance algorithm to the model of matching through institutions. The proof, given in the Appendix, is an adaptation to our model of classical proofs of fairness and strategy-proofness of the Deferred Acceptance algorithm. Theorem 5.1 provides a strong rationale for the use of this mechanism to assign agents to apartments when institutions face distributional constraints.

6. Extensions and discussion

6.1. Multiple institutions

In this subsection we generalize our previous results by allowing households to be attached to multiple institutions.[7] We now assume that the institution assignment mapping τ is a correspondence rather than a function: $\tau : H \to 2^I$, i.e., $\tau(h) \subseteq I$ and $H_i = \{h \in H \mid i \in \tau(h)\}$. We first note that fairness is too demanding and must be weakened when agents can belong to multiple institutions. The following example, inspired by Biró and McDermid [8], illustrates why fairness may fail.

Example 6.1. (**No fair assignments when households belong to multiple institutions**). Consider a market $H = \{h_1, h_2\}$, $I = \{i_1, i_2\}$ and $A = \{a_1, a_2\}$. Let $A_1 = \{a_1\}, A_2 = \{a_2\}, H_1 = H_2 = H$. Apartment preferences $\succ$, households' preferences and apartments' priorities are given by

$\succ_{i_1}$	$\succ_{i_2}$		P_{h_1}	P_{h_2}		π_{a_1}	π_{a_2}
(h_1, a_1)	(h_1, a_2)		a_1	a_1		i_1	i_2
(h_2, a_1)			a_2			i_2	

If each institution assigns one apartment, there is only one assignment that satisfies the distributional constraints:

$$\mu = \begin{pmatrix} h_1 & h_2 \\ a_2 & a_1 \\ i_2 & i_1 \end{pmatrix}.$$

Now, note that $a_1 P_{h_1} a_2$, $\{(h_1, a_1)\} \succ_{i_1} \{(h_2, a_1)\}$ and $i_1 \pi_{a_1} i_2$. Consequently, in the assignment μ we have that pair (h_1, i_1) has justified envy over the pair (h_2, i_2) at apartment a_1.

Notice that the over-demand condition is satisfied in Example 6.1. Hence the inexistence of assignment satisfying fairness and distributional constraints comes from another source, namely the fact that household h_1 belongs to the list of the two institutions which have different preferences over the apartment matched to household h_1. In order to preclude this phenomenon, we do not allow for envy involving households attached to two different institutions and consider the weaker requirement of fairness over households of the same institution. We can then adapt Theorem 5.1 to show that the NDAI still satisfies the desirable properties.

Theorem 6.1. *Consider a matching market with institutions* $E =< I, A, H, \alpha, \tau, \mathbf{R}, \succ, \pi >$ *that satisfies the over-demand condition where households can belong to many institutions.*

1. *The* μ^{NDAI} *is individually rational, non-wasteful, respects distributional constraints and there is no justified envy over households of the same institution; namely, the assignment* μ^{NDAI} *is feasible and fair over households of the same institution.*
2. *There is no fair over households of the same institution assignment that Pareto dominates* μ^{NDAI}.
3. *The NDAI is strategy-proof for households and satisfies the participation criterion.*

[7] This is the case for applications to social housing in Paris. Some households appear on the list of different institutions

6.2. *Flexible quotas*

In this subsection, we discuss the model when institutions face flexible rather than exact quotas. Suppose that, instead of receiving exactly the same number of apartments as the one it initially owns, an institution faces a lower bound and/or an upper bound. On the one hand, this relaxation of the distributional constraints may result in existence of feasible assignments. The nonexistence of feasible assignments in Examples 3.1, 3.2 and of a feasible and fair assignment in Example 3.3 would vanish if institutions faced flexible instead of exact quotas.

On the other hand, as is well known from the literature on controlled school choice (see for example [30]), flexible quotas may increase the degree of freedom of players, and result in nonexistence of fair assignments when fair assignment exist under exact quotas. We illustrate this point in the following example.

Consider a market with two apartments, $A = \{a_1, a_2\}$, three institutions, $I = \{i_1, i_2, i_3\}$, three households $H = \{h_1, h_2, h_3\}$. Let $A_1 = \{a_1\}, A_2 = \{a_2\}$, $H_1 = \{h_1\}, H_2 = \{h_2\}$ and $H_3 = \{h_3\}$. Priorities and preferences are given as follows:

$\succ_{i_1}$	$\succ_{i_2}$	$\succ_{i_3}$
(a_2, h_1)	(a_1, h_2)	(a_2, h_3)
$\varnothing$	$\varnothing$	$\varnothing$

P_{h_1}	P_{h_2}	P_{h_3}
a_2	a_1	a_2
$\varnothing$	$\varnothing$	$\varnothing$

π_{a_1}	π_{a_2}
i_1	i_2
i_3	i_3
i_2	i_1

One might wish to give degree of freedom to accommodate households by setting lower and upper quotas; the approach actually makes the existence of a fair matching more difficult. Suppose that the upper and lower bounds are given by $\{0, 1\}$, $\{0, 1\}$, $\{0, 1\}$ for the three institutions. The reader can verify that there is no feasible, fair assignment with these flexible quotas. On the other hand, suppose that quotas are exact, so that institutions 1 and 2 are assigned one apartment and institution 3 receives no apartement. Then it is easy to check that there exists a fair matching given by

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & \varnothing \\ h_1 & h_2 & \varnothing & h_3 \\ a_2 & a_1 & \varnothing & \varnothing \end{pmatrix}.$$

6.3. *The case of regional preferences*

Kamada and Kojima [4] analyze a matching market with distributional constraints inspired by the assignment of doctors to hospitals in Japan. Instead of considering a simple many-to-one matching between hospitals and residents, they allow for some flexibility in the way hospitals fill their quotas of positions. They introduce "regional preferences" over vectors that indicate the number of residents assigned to each hospital in the region.

In the market with distributional constraints and regional preferences doctors are interested in hospitals, hospitals care about doctors, and regions care about the number of doctors that each hospital can accept. Doctors, regions, and hospitals play a similar role as households, institutions, and apartments in a matching market with institutions. However, there are two main differences between the market with regional preferences and the market with institutions. First, hospitals have priorities over doctors, whereas objects have priorities over institutions in our model. Second, and more importantly, regional preferences in Kamada and Kojima [4] are defined over the distribution of doctors over hospitals as measured by a vector of capacities whereas we consider that institutions care about the specific apartment assigned to households.

Notwithstanding the previous differences, the NDAI algorithm can be applied to produce stable assignments in markets with distributional constraints and regional preferences. In Appendix D, we formally discuss the connection between the two models and how the NDA applies to the market with regional preferences. Our argument relies on noticing that the outer loop generates a set of possible doctor-hospital assignments which naturally relates to a set of capacity vectors; so, the previous capacity vectors drive the regions' intervention during the inner loop. In contrast, the Generalized Flexible Deferred Acceptance algorithm (GFDA), that deals with distributional constraints and regional preferences in [4], cannot be applied to solve the matching through institutions problem. The GFDA deals with regional preferences by updating the capacity vector of each region, i.e., the number of doctors that each hospital can accept at each step. However, the GFDA mechanism does not consider the possibility that different regions can be associated to the same hospital, a possibility which arises in our model when different institutions apply for the same apartment. In other words, regions cannot pool hospitals the way that institutions pool apartments in our model. This difference shows that the GFDA cannot be applied to matching through institutions.

7. Concluding Remarks

We model a matching market with institutions as a three-sided market. Institutions have agents attached to them, and have preferences over the assignment of objects to their agents. Agents have preferences over objects. Objects have priorities over institutions. We show that fair assignments satisfying distributional constraints may fail to exist, and propose a sufficient condition – the over-demand condition – under which we prove existence. Existence derives from the construction of a new algorithm, the Nested Deferred Acceptance (NDA) algorithm, which combines a one-to-one matching between agents and objects and a one-to-many matching between objects and institutions. If interrupters are eliminated from the priority list, as in Kesten [3], the NDA algorithm produces an assignment which is fair, Pareto optimal among fair assignments and strategy-proof for agents. The algorithm can be adapted to allow agents to belong to several institutions.

The NDA algorithm is a general algorithm which can be used in any three-sided market. For example, it may be used to study the assignment of doctors to hospitals and specialties when regions have preferences over capacities as in [4]. It can also be used in the inter-district school problem studied by Hafalir, Kojima and Yenmez [23] where the three sides of the market are students, schools and districts.

Finally, in a companion paper, we explore the application of the NDA algorithm to a three-sided market design problem among scholars, universities and disciplines in the context of the junior chairs in Mexican public universities created in 2013 by the National Council for Research and Innovation (CONACYT) ([31]).

[1] APUR, L' accès au logement social à paris, Tech. rep., Atelier Parisien d'Urbanisme (2014).

[2] U. M. Dur, U. Ünver, Two-sided matching via balanced exchange, Journal of Political Economy 127 (3) (2019) 000–000.

[3] O. Kesten, School choice with consent, The Quarterly Journal of Economics 125 (3) (2010) 1297–1348.

[4] Y. Kamada, F. Kojima, Accommodating various policy goals in matching with constraints, Japanese Economic Review (1) (2019) 000–000.

[5] M. Goto, F. Kojima, A. Tamura, M. Yokoo, Designing matching mechanisms under general distributional constraints, American Economic Journal: Microeconomics 9 (2) (2017) 226–62.

[6] A. Alkan, Nonexistence of stable threesome matchings, Mathematical Social Sciences 16 (2) (1988) 207–209.

[7] C.-C. Huang, Two's company three's a crowd: Stable families and threesome roommate problems, in: European Symposium on Algorithms, Springer, 2007, pp. 558–569.

[8] P. Biró, E. McDermid, Three-sided stable matching with cyclic preference and the kidney exchange problem, Algorithmica 58 (1) (2010) 5–18.

[9] L. Farczadi, K. Georgiou, J. Könemann, Stable marriage with general preferences, Theory of Computing Systems 59 (4) (2016) 683–699.

[10] F. Zhang, et al., Three-dimensional stable matching with hybrid preferences, Journal of Combinatorial Optimization (2017) 1–7.

[11] F. Kojima, School choice: Impossibilities for affirmative action, Games and Economic Behavior 75 (2) (2012) 685–693.

[12] I. E. Hafalir, M. B. Yenmez, M. A. Yildirim, Effective affirmative action in school choice, Theoretical Economics 8 (2) (2013) 325–363.

[13] S. D. Kominers, T. Sönmez, Matching with slot-specific: Theory, Theoretical Economics 11 (2) (2016) 683–710.

[14] A. Abdulkadiroğlu, College admission with affirmative action, Internatonal Journal of Game Theory 33 (4) (2003) 535–549.

[15] A. Abdulkadiroğlu, T. Sönmez, School choice: A design mechanism approach, American Economic Review 93 (3) (2003) 729–747.

[16] L. Ehlers, I. E. Hafalir, M. B. Yenmez, M. A. Yildirim, School choice with controlled choice constraints: Hard bonus versus soft bonus, Journal of Economic Theory 153 (2014) 648–683.

[17] D. Fragiadakis, P. Troyan, Market design under distributional constraints: Diversity in school choice and other applications, Theoretical Economics 12 (12) (2017) 863–908.

[18] I. Bó, Fair implementation of diversity in school choice, Games and Economic Behavior 97 (2016) 54–63.

[19] F. Echenique, M. B. Yenmez, How to control controlled school choice, American Economic Review 105 (8) (2015) 2679–94.

[20] A. Erdil, T. Kumano, Prioritizing diversity in school choice, working paper (2014).

[21] P. Biró, F. Klijn, S. Pápai, Circulation under responsive preferences, in: 1st Catalan Economic Society Conference (2017), 2017, mimeo.

[22] T. Nguyen, R. Vohra, Near feasible stable matching with couples, American Economic Review 108 (11) (2018) 3154–3169.

[23] I. E. Hafalir, F. Kojima, M. B. Yenmez, Interdistrict school choice: A theory of student assignment, Available at SSRN 3307731.

[24] D. Delacrétaz, S. D. Kominers, A. A. Teytelboym, Refugee resettlement, working paper (2016).

[25] J. Leshno, Dynamic matching in overloaded systems, working paper (2014).

[26] F. Bloch, D. Cantala, Dynamic allocation of objects to queuing agents, American Economic Journal: Microeconomics 9 (1) (2017) 88–122.

[27] J. Schummer, Influencing waiting lists, Tech. rep., Working Paper, Kellogg School of Management (2016).

[28] N. Thakral, Matching with stochastic arrival, in: Proceedings of the Sixteenth ACM Conference on Economics and Computation, 2015, p. 343.

[29] D. Gale, L. Shapley, College admision and the stability of marriage, American Mathematical Monthly 69 (1) (1962) 9–15.

[30] L. Ehlers, School choice with control (2010).

[31] D. Cantala, D. Gibaja, The allocation of cátedras conacyt, working paper (2019).

[32] A. E. Roth, The economics of matching: Stability and incentives, Mathematics of Operations Research 7 (4) (1982) 617–628.

[33] J. W. Hatfield, P. Milgrom, Matching with contracts, American Economic Review 95 (4) (2005) 913–935.

[34] J. W. Hatfield, F. Kojima, Substitutes and stability for matching with contracts, Journal of Economic Theory 145 (5) (2010) 1704–1723.

[35] J. W. Hatfield, S. D. Kominers, Matching in networks with bilateral contracts, American Economic Journal Microeconomics 4 (1) (2012) 176–208.

Appendix

Appendix A. The NDA algorithm

Initialization

Consider a market $< I, A, H, \alpha, \tau, \mathbf{R}, \succ, \pi >$. The assignment is initialized to be the empty assignment, so $\mu^0(i) = \mu^0(a) = \mu^0(h) = \varnothing$, i.e. $\theta^0(i) = \theta^0(a) = \varphi^0(h) = \varnothing$ for all $i \in I$, $a \in A$, $h \in H$.

Let $A_h^t = A$ and $t := 1$.

A^t. Eliciting the demand of households (the outer loop)

All unassigned households h ask for their most preferred apartment in A_h^t, denoted by D_h^t, while matched households h' iterate their demand to their match, i.e. $D_{h'}^t = \{\varphi_A^{t-1}(h')\}$.

For all $i \in I$ and $a \in A$, we define the set of households that belong to institution i and demand apartment a as follows:

$$H_{a,i}^t = \{h \in H \mid D_h^t = \{a\} \text{ and } i = \tau(h)\}.$$

The set of pairs (a, h) that can be assigned to institution i is defined as

$$\mathcal{M}_i^t = \{(a, h) \in A \times H \mid (a, h) \succ_i \varnothing \text{ and } h \in H_{a,i}^t\}.$$

B^t. Iteration over $\mathcal{M}_\mathbf{i}^\mathbf{t}$ to match institutions and apartments (the inner loop)

Let $\theta^s(i) = \varnothing$ for all $i \in I$, and $\tilde{\mathcal{M}}_i^s = \mathcal{M}_i^{t,s}$, $s := 1$. We let $\tilde{\mathcal{M}}_i^s$ denote the choice set of institution i at the s iteration of the inner loop and omit the index of the iteration of the outer loop.

B^t.1 All institutions i demand the set of pairs $Ch_i(\tilde{\mathcal{M}}_i^s, m_i)$. So, the set of institutions that demand an apartment a is

$$I_a^s = \{i \in I \mid \text{ there exists } (a, h) \in Ch_i(\tilde{\mathcal{M}}_i^s, m_i)\}.$$

B^t.2 For all apartments a such that $I_a^s \neq \varnothing$, apartment a is assigned to the institution with the highest priority under π_a, i.e. $a \in \theta^s(i)$ if and only if $i = \max_{\pi_a} I_a^s$.

For all institutions i, let $\tilde{\mathcal{M}}_i^{s+1} := \tilde{\mathcal{M}}_i^s \setminus \{(a, h) \in \tilde{\mathcal{M}}_i^s \mid (a, h) \in Ch_i(\tilde{\mathcal{M}}_i^s, m_i) \text{ and } a \notin \theta^s(i)\}$. That is to say, we delete from the set $\tilde{\mathcal{M}}_i^s$ those pairs containing an apartment which rejected institution i.

If $|\theta^s(i)| = m_i$ for all institutions i, or $\tilde{\mathcal{M}}_i^{s+1} = \varnothing$ for all institutions i for which $|\theta^s(i)| < m_i$, go to B^t.3; otherwise, let $s = s + 1$ and go to B^t.1.

B^t.3 Rename $\theta^t(i) := \theta^S(i)$, where S is the last iteration of B^t.1. Furthermore, each pair (a, i) is tentatively assigned to household h if and only if $a \in \theta^t(i)$ and $(a, h) \in Ch_i(\tilde{\mathcal{M}}_i^S, m_i)$. That is to say $\varphi^t(h) = (a, i)$.

C^t. Iteration over $\mathbf{D}_\mathbf{h}^\mathbf{t}$ For all unassigned households h, let $A_h^{t+1} := A_h^t \setminus \{\max_{P_h} A_h^t\}$. If each household has been rejected by all the apartments in her preference list or is matched, the tentative assignment becomes the final assignment. Otherwise, $t := t + 1$, and go to A^t.

Appendix B. Example Details

Details of example 5.1:

Running the NDA algorithm, the elicited demand of households at step 1 are described in Table B.1. Table B.2 shows the institutions that demand each apartment, with apartment a_1 demanded by both institutions.

I	$H^1_{a_1,i}$	$H^1_{a_2,i}$	$H^1_{a_3,i}$	$\mathcal{M}^1_i = \tilde{\mathcal{M}}^1_i$
i_1	h_1, h_6	h_2, h_5	$\varnothing$	$(a_1, h_1), (a_2, h_2)$
i_2	h_3, h_4, h_7	$\varnothing$	$\varnothing$	$(a_1, h_3), (a_1, h_4), (a_1, h_7)$

Table B.1: A^1. Elicited Demand of Households.

Note that $i_2\pi_{a_1}i_1$. Phase B^1 stops in one step. Hence, the tentative assignment produced at the end of step 1 is given in the last column of Table B.3.

A	I^1_a
a_1	i_1, i_2
a_2	i_2
a_3	$\varnothing$

Table B.2: B^1. Institutions demand.

I	$Ch^1_i(\tilde{\mathcal{M}}^1_i, m_i)$	μ^1
i_1	$(a_1, h_1), (a_2, h_2)$	(a_2, h_2)
i_2	(a_1, h_3)	(a_1, h_3)

Table B.3: B^1. Iteration over the sets $\mathcal{M}^1_i$.

The algorithm continues to step 2, where Table B.4 shows the demands of households at this step, and Table B.5 the institutions that demand each apartment. Notice that institution i_2 has a higher priority than institution i_1 under priorities π_{a_1} and π_{a_2}. Moreover, we have that $m_2 = i_1$ and $(a_2, h_4) \succ_{i_1} (a, h)$ for all $(a, h) \in \mathcal{M}^2_2$. So, the tentative assignment produced at the end of step 2 is μ^2, which is shown in the last column of Table B.6.

I	$H^2_{a_1,i}$	$H^2_{a_2,i}$	$H^2_{a_3,i}$	$\mathcal{M}^2_i = \tilde{\mathcal{M}}^1_i$
i_1	h_5	h_2, h_6, h_1	$\varnothing$	$(a_2, h_1), (a_2, h_2)$
i_2	h_3	h_4, h_7	$\varnothing$	$(a_1, h_3), (a_2, h_4), (a_2, h_7)$

Table B.4: A^2. Elicited Demand of Households.

It is important to note that the assignments (h_3, a_1, i_2) and (h_2, a_2, i_1) are disrupted at the end of step 2 because $i_2\pi_{a_1}1$ and $(a_2, h_4) \succ_{i_2} (a_1, h_3)$. Households h_2 and h_3 are rejected from apartments a_2 and a_1, respectively.

The algorithm continues to step 3 because not all households have been rejected by all their acceptable apartments. For example households h_2 and h_3 have not been rejected from the apartments a_1 and a_2, respectively. The demand of households at step 3 is shown in Table B.7.

A	I_a^2
a_1	$1,2$
a_2	$1,2$
a_3	$\varnothing$

Table B.5: B^2. Elicited demand of institutions.

I	$Ch_i^2(\tilde{\mathcal{M}}_i^1, m_i)$	μ^2
i_1	(a_2, h_1)	$\varnothing$
i_2	(a_2, h_4)	(a_2, h_4)

Table B.6: B^2. Iteration over the sets $\mathcal{M}_i^2$.

From Table B.7, we observe that each apartment is demanded by a different institution. So, the tentative assignment produced at the end of step 3 is shown in the last column of Table B.8.

I	$H^3_{a_1,i}$	$H^3_{a_2,i}$	$H^3_{a_3,i}$	$\mathcal{M}_i^3 = \tilde{\mathcal{M}}_i^1$
i_1	h_2	$\varnothing$	h_1, h_5, h_6	$(a_1, h_2), (a_3, h_1), (a_3, h_6)$
i_2	$\varnothing$	h_3, h_4	h_7	$(a_2, h_3), (a_2, h_4), (a_3, h_7)$

Table B.7: A^3. Elicited Demand of Households.

I	$Ch_i^3(\tilde{\mathcal{M}}_i^1, m_i)$	μ^3
i_1	$(a_1, h_2), (a_3, h_1)$	$(a_1, h_2), (a_3, h_1)$
i_2	(a_2, h_3)	(a_2, h_3)

Table B.8: B^3. Iteration over the sets $\mathcal{M}_i^3$.

At the end of step 3, we note that h_5, h_6 and h_7 have been rejected from all their acceptable apartments. However, h_4 is rejected by a_2, and her last acceptable apartment is a_3. Thus, the algorithm continues to step 4, where h_4 demands the apartment a_3, and households h_1, h_2, h_3 iterate their demand to their match. Institutions i_1 and i_2 demand the apartment a_3 as $(a_3, h_1) \succ_{i_1} \varnothing$ and $(a_3, h_4) \succ_{i_2} \varnothing$. However, we know that institution i_1 has a higher priority than institution i_2 under the priority π_{a_3}, which implies that household h_4 is rejected from the apartment a_3.

Therefore, the NDA algorithm stops at the end of step 4, and produces the assignment

$$\mu^{NDA} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_3 & a_1 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ i_1 & i_1 & i_2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

Details of example 5.2:

Given that Example 5.1 corresponds to stage 0.1 of the NDAI, we delete all the pairs (a_1, h) from the

I	$H^{i1}_{a_1}$	$H^{i1}_{a_2}$	$H^{i1}_{a_3}$
i_1	h_1, h_6	h_2, h_5	$\varnothing$
i_2	$h_3,\ h_4, h_7$	$\varnothing$	$\varnothing$

Table B.9: A[1]. Elicited Demand of Households, Stage 1.1

priority $\succ_{i_2}$ at Stage 0.2. We get

$$\succ^1_{i_1} = \succ_{i_1} \quad \text{and} \quad \begin{array}{c} \succ^1_{i_2} \\ \hline (a_2, h_3) \\ (a_2, h_4) \\ (a_2, h_7) \end{array}$$

Consider stage 1.1. We run the NDA algorithm with priorities $\succ^1$. Step 1.1 of this NDA algorithm is summarized in Table B.9. Given that $m_1 = i_2$, and $i_1 \pi_{a_1} i_2$, the tentative assignment is

$$\mu^1 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ i_1 & i_1 & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

The NDA in Stage 1.1 moves to step 2. Phase A[2] of the algorithm is illustrated in Table B.10. Following the institutions' preferences and the fact that $i_2 \pi_{a_3} i_1$, the tentative assignment produced at the end of step 2 is

$$\mu^2 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & \varnothing & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ i_1 & \varnothing & i_2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

I	$H^2_{a_1,i}$	$H^2_{a_2,i}$	$H^2_{a_3,i}$
i_1	h_1, h_5	h_2, h_6	$\varnothing$
i_2	$\varnothing$	h_3, h_4, h_7	$\varnothing$

Table B.10: A[2]. Elicited Demand of Households.

The NDA algorithm moves to step 3. We display the demands of households in Table B.11. Since institution i_1 has a higher priority than institution i_2 under π_{a_3}, we get the tentative assignment μ^3.

$$\mu^3 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & \varnothing & a_2 & \varnothing & a_3 & \varnothing & \varnothing \\ i_1 & \varnothing & i_2 & \varnothing & 1 & \varnothing & \varnothing \end{pmatrix}.$$

Note that household h_2 has not been rejected by the apartment a_3, her last acceptable apartment. The NDA moves to step 4 and the demands of households are shown in Table B.12.

I	$H^3_{a_1,i}$	$H^3_{a_2,i}$	$H^3_{a_3,i}$
i_1	h_1, h_2	$\varnothing$	h_5, h_6
i_2	$\varnothing$	h_3	h_4, h_7

Table B.11: A^3. Elicited Demand of Households.

I	$H^4_{a_1,i}$	$H^4_{a_2,i}$	$H^4_{a_3,i}$
i_1	h_1	$\varnothing$	h_2, h_6
i_2	$\varnothing$	h_3	$\varnothing$

Table B.12: A^4. Elicited Demand of Households.

We observe that all households have been accepted or rejected at the end of the step 4. There are no interrupters because no apartment is rejected by any institution. Therefore, the NDAI algorithm stops and produces the following assignment.

$$\mu^{NDAI} = \mu^4 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & a_3 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ i_1 & i_1 & i_2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

□

Appendix C. Proofs

Proof of Proposition 4.1:

Let π be a profile of priorities when institutions pool apartments. To prove that the NDA mechanism satisfies the participation criterion, we proceed by contradiction. Assume that

$$\varphi_A^{NDA}[\tilde{\pi}](h) P_h \varphi_A^{NDA}[\pi](h)$$

for some $h \in H$.

Consider that $\varphi^{NDA}[\tilde{\pi}](h) = (a, i)$, and $\varphi^{NDA}[\pi](h') = (a, j)$. We analyze the following two cases.

Case 1. $i \neq j$. Note that institution i owns the apartment a since $a \in \theta^{NDA}[\tilde{\pi}](i)$ and no institution pools its apartments under $\tilde{\pi}$. Thus, i has top priority at π_a, which implies that $i \pi_a j$. Now, notice that under priorities profile π, agent h must have demanded apartment a at some step t. At that step, institution i must have demanded apartment a at some step t (either for h or for some other agent in H_i). Moreover, given that the apartment a is finally assigned to $h' \in H_j$, institution j must have demanded apartment a at some state t' and obtained it. But as institution i has a priority over institution j for apartment a, this results in a contradiction.

Case 2. $i = j$, i.e., households h and h' belong to the same institution. We analyze the the following two cases

Case 2.1 $aP_{h'}\varphi_A[\tilde{\pi}](h')$, which implies that a is acceptable for households h and h'. Then, h and h' demand apartment a at steps t and t' during Phase A of the $NDA[\tilde{\pi}]$ procedure. Moreover, we have that $(h, a) \in \mu^{NDA}[\tilde{\pi}](i)$ and $(h', a) \notin \mu^{NDA}[\tilde{\pi}](i)$; together with the fact that the preference $\succ_i$ is responsive, this implies that

$$(h, a) \succ_i (h', a). \tag{C.1}$$

By applying the previous reasoning to the $NDA[\pi]$ assignment procedure, we obtain

$$(h', a) \succ_i (h, a) \tag{C.2}$$

because $(h', a) \in \mu^{NDA}[\pi](i)$ and $\varphi_A^{NDA}[\pi](h) \neq a$, a contradiction.

Case 2.2 $\varphi_A[\tilde{\pi}](h')P_{h'}a$, and consider $\varphi_A[\tilde{\pi}](h') = a'$. Given that $a'P_{h'}a$ and $\varphi_A^{NDA}[\pi](h') = a$, there exists households $h'' \neq h$ such that $\varphi^{NDA}[\pi](h'') = (a', j')$. As before, we have that $j' = i$ or not.

If $j' \neq i$, this situation is analogous lo the Case 1, which means that a contradiction arises due to $i\pi_{a'}j$. In the other case, we can apply the same reasoning of Case 2 on h' to h'', i.e., we will end up generating a contradiction by assuming that $a'P_{h''}\varphi_A(h'')$; otherwise, an infinite sequence of households $\{h^{(k)}\}$, which is not possible since H is finite.

Therefore, no household h is worse off under $\mu^{NDA}[\pi]$, which implies that the NDA algorithm satisfies the participation criterion.

Proof of Lemma 1:

Consider a step t of the NDA algorithm. We know that phase B^t starts with the set of acceptable apartment-households pairs $\tilde{\mathcal{M}}_i^1 = \mathcal{M}_i^t$. This implies that each institution i demands the set $Ch_i(\tilde{\mathcal{M}}_i^t, m_i)$.

Let $A_i^t = \{a \in A | (a, h) \in \mathcal{M}_i^t\}$ be the set of apartments demanded by some household that belong to institution i at step t. Since $|A_i^t| \leq A$, there are no distributional constraints and priorities are responsive, all pairs (a, h) at the top of the institution's preferences belong to the set $Ch_i(\tilde{\mathcal{M}}_i^1, m_i)$. Each institution i demands all the apartments in the set A_i^t. So, all the apartments in $\bigcup_{i \in I} A_i^t$ belong to $\theta^1(j)$, for some $j \in I$, at the end of Phase B^t.2.

Hence

$$\tilde{\mathcal{M}}_i^2 = \tilde{\mathcal{M}}_i^1 \setminus \{(a, h) \in \tilde{\mathcal{M}}_i^1 | a \in \theta^1(j) \text{ for some } j \neq i\} = \tilde{\mathcal{M}}_i^1 \setminus \tilde{\mathcal{M}}_i^1 = \varnothing$$

for all institution i. Therefore, phase B^t stops in one iteration.

Proof of Lemma 2: Consider an apartment a that is assigned by some institution i at some step t, i.e. a belongs to $\theta^t(i)$. Since households iterate their demand to their match, we have that $i \in I_a^{t+1}$. In other words, this apartment is demanded by some institution at step $t + 1$. Since there are no distributional constraints and priorities are responsive, the apartment a is assigned to some institution at the end of step $t + 1$ (the institution in I_a^{t+1} with the highest priority at π_a). Iterating this argument, we conclude that the apartment a is assigned to some institution at all steps $t' \geq t$.

Therefore $\mu^{NDA}(a) \neq \varnothing$ because the NDA algorithm stops in a finite number of steps.

Proof of Theorem 4.1:

Individual Rationality. For all institutions $i \in I$, we know that $\mu^{NDA}(i) \subseteq Ch_i(\mathcal{M}_i^T, m_i)$, where T is the last iteration of the NDA algorithm. Thus, $(a,h) \succ_i \varnothing$ for all $(a,h) \in \mu^{NDA}(i)$. Therefore, $\mu^{NDA}(i) \succ \varnothing$ for all $i \in I$.

Moreover, the NDA algorithm stops when every unmatched household has been rejected by all her acceptable apartments, in this case $\varphi(h) = \varnothing$, or every household is matched to some acceptable apartment, i.e. $\varphi_A(h) P_h \varnothing$ for all $h \in H$.

Non-wastefulness. We proceed by contradiction. We assume the existence of a household-institution pair (h,i) that claims an empty apartment a. Hence, we must have: i) $aP_h\varphi(h)$, ii) $(a,h) \in Ch_i(\mu^{NDA}(i) \cup \{(a,h)\}, m_i)$, and iii) $\theta^{NDA}(a) = \varnothing$. Condition i) implies that household h demands the apartment a at some step t of the NDA algorithm and condition ii) guarantees that the pair (a,h) is acceptable for institution i. Thus, institution i demands the apartment a at step t. The apartment must then be assigned at step t and, applying Lemma 2, it must remain assigned at step T, contradicting condition (iii).

There is no justified envy. Consider, to the contrary, the existence of a pair (h,i) that has justified envy over a pair (h',i'), where $\tau(h) = i$ and $\tau(h') = i'$. Then, there exists an apartment a such that $\varphi^{NDA}(h') = a$, $a \in \theta(i')$, and

i. $aP_h\varphi^{NDA}(h)$,

ii. $(a,h) \in Ch_i(\mu^{NDA}(i) \cup \{(a,h)\}, m_i)$,

iii. $i\pi_a i'$.

By condition i), household h demands the apartment a at some step $\bar{t}$. Moreover, condition ii) ensures that the pair (a,h) is acceptable for institution i,so that we have $i \in I_a^{\bar{t}}$. We distinguish the following cases.

Case 1. $i = i'$, i.e. $\tau(h) = \tau(h') = i$. Since $\varphi^{NDA}(h') = a$, household h' must demand the apartment a at some step t'. Furthermore, $(a,h') \in Ch_i(\mathcal{M}_i^{t'}, m_i)$ because (a,h') belongs to the set $\mu^{NDA}(i)$. Given that $(a,h) \in \mathcal{M}_i^t$ but $\varphi^{NDA}(h) \neq a$, responsiveness of preferences ensures that

$$\{(a,h')\} \succ_i \{(a,h)\}. \tag{C.3}$$

Moreover, since no apartment can be paired twice, condition ii) implies that

$$\{(a,h)\} \succ_i \{(a,h')\},$$

in contradiction with (C.3).

Case 2. $i \neq i'$, i.e. $\tau(h) \neq \tau(h')$. We know that $a \in \theta^{NDA}(i')$, which implies the existence of some step t' where

$$i'\pi_a j \text{ for all } j \in I_a^t, \text{ for all } t \geq t'. \tag{C.4}$$

Also, we know that institution i demands apartment a at step $\bar{t}$.

Case 2.1 If $\bar{t} \geq t'$, expression (C.4) implies that

$$i'\pi_a i,$$

in contradiction with condition iii).

Case 2.2 If $\bar{t} < t'$, let i'' be the institution such that $a \in \theta^{t'-1}(i'')$. By the NDA's Phase B, institutions do not get an apartment whenever there is an institution with a higher priority that demands it, which means that i'' is the institution with the highest priority at π_a among the institutions that demanded a from steps 1 to $t'-1$. By the previous discussion, we have that $i''\pi_a i$ if $i'' \neq i$. Moreover, i'' is rejected from a at step t' since expression (C.4) holds. By transitivity, we conclude that $i'\pi_a i$ in contradiction with condition iii).

In all cases we get a contradiction, showing that there is no justified envy at the assignment μ^{NDA}.

Pareto undomination. We proceed as in Gale and Shapley [29]. To prove that μ^{NDA} is Pareto undominated, we show that in any other fair assignment, each household gets the same apartment or an apartment less preferred than $\varphi^{NDA}(h)$.

An apartment a is said to be **achievable** for a household h if there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu(h) = a$. We proceed by induction to show that no household is rejected by an achievable apartment during the NDA algorithm.

Inductive base. We initialize the NDA algorithm at step $t = 0$, where no household has asked for an apartment. Hence, no household has been rejected by any achievable apartment.

Inductive hypothesis. At step t we assume that no household has been rejected by any achievable apartment. In other words, if a household is rejected by some apartment, this apartment is not achievable for her.

Inductive step. Consider that some household h^* is rejected at step $t+1$ from an apartment a. We assume, to the contrary, that a is achievable for household h^*. Thus, there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu(h^*) = a$. Notice that the pair (a, h^*) is acceptable for the institution i^* to whom h^* belongs.

Now, let h be the household assigned to the apartment a at the end of step $t+1$, and i the institution to which h belongs. We distinguish the following cases.

Case 1. $i = i^*$. Since $\varphi^{t+1}(h) = a$, we must have

$$\{(a, h)\} \succ_i \{(a, h^*)\} \tag{C.5}$$

because $(a, h^*) \notin \mu^{t+1}(i)$. Since preferences are responsive, we have that

$$(a, h) \in Ch_i(\mu(i) \cup \{(a, h)\}, m_i).$$

As h prefers a to all the apartments that have not rejected her, by the induction hypothesis, household h prefers a to any other achievable apartment for her:

$$aP_h\varphi^\mu(h).$$

Moreover, $(a, h^*) \in \mu(i)$. Thus the pair (h, i) has justified envy over the pair (h^*, i) at the apartment a in the assignment μ, which contradicts the fact that μ is a fair assignment.

Case 2. $i \neq i^*$. We know that $\varphi^{\mu}(h^*) = (a, i^*)$, i.e. the pair (a, h^*) is acceptable for the institution i^*, so $i^* \in I_a^{t+1}$. Moreover, $i \in I_a^{t+1}$ because $\varphi^{t+1}(h) = (a, i)$. Given that $a \in \theta^{t+1}(i)$, we conclude that

$$i\pi_a i^*. \tag{C.6}$$

By the inductive hypothesis, we know that household h strictly prefers a to any other achievable apartment for her, i.e.

$$aP_h\varphi^{\mu}(h). \tag{C.7}$$

Next, the pair (a, h) must be acceptable for institution i. Moreover, $(a, h^*) \in \theta^{\mu}(i^*)$, so a is not assigned to i at μ. Furthermore, as there are no distributional constraints and institutions' preferences are responsive,

$$(a, h) \in Ch_i(\mu(i) \cup \{(a, h)\}, m_i). \tag{C.8}$$

By C.6, C.7 and C.8, the pair (h, i) has justified envy over the pair (h^*, i^*) at the apartment a in the assignment μ, which contradicts the fact that μ is fair.

In all cases, a contradiction arises when we assume that household h^* is rejected by some achievable apartment a. So, no household is rejected by an achievable apartment. This shows that μ^{NDA} is Pareto undominated among fair assignments.

Truth-telling is a dominant strategy for households. We construct the proof as in Roth [32].

For each household h, we say that P'_h is a **successful** misrepresentation of P_h if P'_h is a preference list such that

$$\varphi^{NDA}[P'_h, P_{-h}](h)P_h\varphi[P](h).$$

Let $a' := \varphi^{NDA}[P'_h, P_{-h}](h)$, we define the preference list P''_h where the apartment a' is declared as the most preferred apartment of h. Let P' and P'' be the preference profiles where household h reports P'_h and P''_h, respectively, and other households do not change their reported preferences. The following lemma establishes that P'_h and P''_h are equivalent in the sense that

$$\varphi^{NDA}[P'_h, P_{-h}](h) = \varphi^{NDA}[P''_h, P_{-h}](h).$$

Lemma 3. *Consider a matching market through institutions with no distributional constraints. Then* $\varphi^{NDA}[P'_h, P_{-h}](h) = \varphi^{NDA}[P''_h, P_{-h}](h)$.

Proof. As noted above, the assignment $\mu^{NDA}[P']$ is fair with respect to P'. Let $i := \tau(h)$. At $\varphi^{NDA}[P'_h, P_h]$, (h, i) has no justified envy over any other pair under P'' because no apartment is preferred to a' under P''_h. Since other preferences did not change between P' and P'', this implies that $\mu^{NDA}[P']$ is fair with respect to the profile P''. Furthermore, apartment a' is the best acceptable achievable apartment of h under P'', and since $\mu^{NDA}[P'']$ is Pareto undominated by any other fair assignment, we conclude that

$$\mu^{NDA}[P'](h) = \mu^{NDA}[P''](h).$$

□

The following Lemma establishes that households are not worse off when a household successfully misrepresents her true preference list.

Lemma 4. *Consider P'_h a preference list different from the true preference list of h and let $P' = (P'_h, P_{-h}), P = (P_h, P_{-h})$. If $\varphi^{NDA}[P'](h)$ is weakly preferred to $\varphi^{NDA}[P](h)$, then for each household $h' \neq h$, either*

$$\varphi^{NDA}[P'](h') P_{h'} \varphi^{NDA}[P](h') \text{ or } \varphi^{NDA}[P'](h') = \varphi^{NDA}[P](h').$$

Proof. We proceed by contradiction. Assume that $aP_{h'}a'$ for some $h' \in H$, where

$$\varphi^{NDA}[P'](h') = a' \text{ and } \varphi^{NDA}[P](h') = a \text{ for } h' \neq h.$$

So, there exists a step t of $NDA[P']$ at which h' is rejected from a. By Lemma 2, the apartment a is ultimately assigned to some household h''. We distinguish the following cases.

Case 1. Consider that $\tau(h'') = \tau(h') = i$. Since $\mu^{NDA}[P']$ is fair, we have that $\{(a, h'')\} \succ_i \{(a, h')\}$. Let $a'' := \varphi^{NDA}[P](h'')$, we disitnguish the following sub-cases.

Case 1.1 $aP_{h''}a''$. Then (h'', i) has justified envy over (h, i) at apartment a in the assignment $\mu^{NDA}[P]$, a contradiction.

Case 1.2 $a''P_{h''}a$. By Lemma 2, there exists a household h''' such that $\varphi^{NDA}[P'](h''') = a''$. So, we apply the previous reasoning on h'' to h''', which will end up generating either a contradiction, or an infinite sequence of households $\{h^{(k)}\}$. But because H is finite, the sequence must end at some step, yielding a contradiction.

Case 2. Consider that $i = \tau(h') \neq \tau(h'') = i'$. Since $\mu^{NDA}[P']$ is fair, we have that $i'\pi_a i$. Let $a'' := \varphi_A^{NDA}[P](h'')$, we distinguish the following sub-cases.

Case 2.1 $aP_{h''}a''$. Then (h'', i') has justified envy over (h, i) at apartment a in the assignment $\mu^{NDA}[P]$, a contradiction.

Case 2.2 $a''P_{h''}a$. By Lemma 2, there exists a household h''' such that $\varphi_A^{NDA}[P'](h''') = a''$. So, we apply the previous reasoning on h'' to h''', which will end up generating either a contradiction, or an infinite sequence of households $\{h^{(k)}\}$. But because H is finite, the sequence must end at some step, yielding a contradiction.

We conclude that no household h' is worse off under assignment $\mu^{NDA}[P']$. □

We return to the main body of the proof. Assume by contradiction the existence of a household h^* and a misrepresentation P'_{h^*} of P_{h^*} such that,

$$a'P_{h^*}a,$$

where $\varphi^{NDA}[P](h^*) = a$ and $\varphi^{NDA}[P'_{h^*}, P_{-h^*}] = a'$. By Lemma 3, we can replace the misrepresentation by a misrepresentation P' where household h puts apartment a' on the top of her preference list.

As in Roth [32], we say that household h **makes a match** at step t of the NDA algorithm, if h demands $\varphi^{NDA}(h)$ at step t. Our first claim shows that if a household misrepresents her preferences, any household which is unmatched at the true preferences remains unmatched after the manipulation.

Claim 1. Consider household h^* with preferences P_{h^*} and P'_{h^*} such that

$$\varphi^{NDA}[P'_{h^*}, P_{-h^*}](h^*) R_{h^*} \varphi^{NDA}[P](h^*).$$

If $\varphi^{NDA}[P](h^0) = \varnothing$ then $\varphi_A^{NDA}[P'](h^0) = \varnothing$.

Proof. By contradiction, consider that h^0 gets an apartment at P'. Since assignments $\mu^{NDA}[P]$ and $\mu^{NDA}[P']$ are non-wasteful, this means that some household h, previously matched at P, is unmatched at P', violating Lemma 4. □

The proof now runs by induction on the steps of the algorithm. We first show that if a household makes a match in the last step of the NDA algorithm under the true preferences, she cannot successfully manipulate her preferences.

Claim 2. Suppose that h^* makes a match at t^*, with $1 \leq t^* \leq T$, then $\varphi^{NDA}[P'](h) = \varphi^{NDA}[P](h)$ for all h that makes a match at T. Moreover, if h^* makes a match at T, there is no profitable deviation P'_{h*} of her true preference list P_{h^*}.

Proof. We first present the argument for $h \neq h^*$. Let T be the last step of the $NDA[P]$ and suppose that household h makes a match at step T, say $a = \varphi_A^{NDA}[P](h)$. Since $\mu^{NDA}[P]$ is non-wasteful, all apartments are matched, and at $T-1$ either

1. a is unmatched, or
2. a is matched to a household h_1 who is unmatched at $\mu^{NDA}[P]$.

Case 1. Since apartment a was unmatched at $T-1$, all matched households prefer their match at $\mu^{NDA}[P]$ to a. By Lemma 4, this implies that none of them gets a at $\mu^{NDA}[P'_{h^*}, P_{-h^*}]$. By Lemma 1 all unmatched households are still unmatched. So, by non-wastefulness, h gets a and does not strictly improve her match under the profile (P'_{h^*}, P_{-h^*}).

Case 2. Let h_1 be the household matched with a at $T-1$, who becomes unmatched at T. For any matched $h_2 \neq h_1$, who prefer a to $\varphi^{NDA}[P](h_2)$, we must have either $\{(a, h_1)\} \succ_{\tau(h_1)} \{(a, h_2)\}$, (by fairness of $\mu^{NDA}[P]$) , or $\tau(h_1) \pi_a \tau(h_2)$. Thus, if h strictly improves under (P'_{h^*}, P_{-h^*}) and a becomes available, either h_1 or an unmatched household, gets a in $\mu^{NDA}[P'_{h^*}, P_{-h^*}]$ because the assignment is fair, a contradiction to Claim 1.

In both cases we conclude that $\varphi_A^{NDA}[P'_{h^*}, P_{h^*}](h) = \varphi_A[P](h)$. The same argument applies to $h = h^*$ since other households do not improve their assignment under (P'_{h^*}, P_{-h^*}). □

Next, consider that h^* makes a match at some step t^* of the $NDA[P]$ procedure, with $1 \leq t^* < T$. We show that no household, matched after t^*, changes its final assignment when household h^* misrepresents her preferences.

Claim 3. Suppose that h^* makes a match at t^*, $1 \leq t^* \leq T$ and P'_{h^*} and P_{h^*} are such that

$$\varphi^{NDA}[P'_{h^*}, P_{-h^*}](h^*) R_{h^*} \varphi^{NDA}[P](h^*).$$

Then $\varphi^{NDA}[P'_{h^*}, P_{-h^*}](h^t) = \varphi^{NDA}[P](h^t)$ for $h^t \neq h^*$ who makes a match at t, where $t^* \leq t \leq T$.

Proof. The proof is by induction.

Base of induction. At $t = T$, the statement is true by Claim 2.

Inductive hypothesis. Suppose that the property holds for all steps $t+1, .., T$

Inductive step. Let a^t be the match of h^t at $\varphi^{NDA}[P](h^t)$.

Case 1. a^t is unmatched at $t-1$. Since a^{t-1} is unmatched at t, all households matched before t or at t strictly prefer their match to a^{t-1}. By Lemma 4 they do not get a^{t-1} at P'. By the inductive hypothesis and Claim 1, all households who make a match after t either get the same apartment or nothing, so that $\varphi^{NDA}[P'_{h^*}, P_{-h^*}](h^t) = a^t = \varphi^{NDA}[P](h^t)$.

Case 2. a^t is matched at $t-1$. Let h^{t-1} be the match of a^t at $t-1$. So h^{t-1} is chosen among all households who prefer a^t to their match and make a match before t. By Claim 1, fairness of $\varphi^{NDA}[P']$ and Lemma 4, apartment a^t is assigned either to h^{t-1} or to a household h', at $NDA[P']$, that makes a match after t. However, this is not possible.

Note that, since $a^t = \varphi_A^{NDA}[P](h^t)$, household h^{t-1} is rejected from a at step t, i.e., h^{t-1} makes a match after t. By the hypothesis of induction, we know that $\varphi_A^{NDA}[P'](\overline{h}) = \varphi_A^{NDA}[P](\overline{h})$ for all $\overline{h} \in \{h^{t-1}, h'\}$. Hence $\varphi_A^{NDA}[P'](h') \neq a^t$ because $\varphi_A^{NDA}[P](h^t) = a^t$. Then, by Lemma 4, h^t gets a^t or a better apartment at $\mu^{NDA}[P']$.

Consider that $\varphi_A^{NDA}[P'](h^t) = a P_{h^t} a^t$. Then, there exists h'' such that

$$\varphi^{NDA}[P](h'') = (a, j) \qquad \text{and} \qquad \varphi^{NDA}[P'](h'') = (a', j).$$

We distinguish the following cases:

Case 2.1 $a P_{h''} a'$, then h^t and h'' demand a during the $NDA[P']$ at steps s and s', respectively. We consider the following subcases.

Case 2.1.1 $i = \tau(h^t) = \tau(h')$, then $(h^t, a) \succ_i (h'', a)$, because $\varphi_A^{NDA}[P'](h^t) = a$. However, $\varphi_A[P](h') = a$ and $a P_{h^t} a^t$, which means that h^t asked for a in some step $\overline{t} < t$ of the $NDA[P]$, but it was rejected. Since the NDA outputs a fair assignment, we conclude that $(h, a) \succ_i (h^t, a)$, a contradiction.

Case 2.1.2 $i = \tau(h^t) \neq \tau(h') = j$, then $i \pi_a j$ because $(h^t, a) \in \mu^{NDA}[P'](i)$. Also, we know that $(h'', a) \in \mu^{NDA}[P](j)$, which means that i demanded a in some step $\overline{t} < t$ of the $NDA[P]$, but it was rejected. Hence, $j \pi_a i$ because $\mu^{NDA}[P]$ is fair, which is a contradiction.

Case 2.2 $a' P_{h''} a$, then there exists h''' such that $\varphi_A[P](h''') = a'$, but $\varphi_A[P](h''') \neq a'$. Note that we can apply the same reasoning to generate generate an infinite sequence of households $\{h^k\}$, which is not possible since H is finite.

In any case, it is not possible that h^t gets a better apartment than a^t. Therefore, $\varphi^{NDA}[P'_{h^*}, P_{-h^*}](h^t) = a^t = \varphi^{NDA}[P](h^t)$. ☐

At any step where household h^* makes a match, Claim 3 implies that $\varphi_A^{NDA}[P'](h^*) = \varphi_A^{NDA}[P](h^*)$. Hence there is no successful misrepresentation of P_{h^*}, and the NDA is strategy-proof.

Proof of Theorem 5.1 : Let x^* be the last iteration of the NDAI mechanism, i.e. there are no interrupter at the end of the NDA phase $x^*.1$. The last step of the NDA phase is denoted by T.

Individual Rationality. Follows from the same argument as in Theorem 4.1.

Distributional Constraints (feasibility). Consider an institution i that does not fulfill its quota in the assignment μ^{NDAI}. Thus, $|\mu^{NDAI}(i)| < m_i$, i.e. there is at least one apartment a that remains unassigned. We know that μ^{NDAI} is Individually rational. By the over-demand condition, for all institutions i , there exists an unassigned household h such that (a, h) is acceptable for i and $aP_h\varnothing$. Now consider the NDA phase at Stage $x^*.1$ If $(a, h) \in \mathcal{M}_i^t$, since i did not fulfill its quota and (a, h) is acceptable for i whose preferences are responsive, there must exist an institution j such that $j\pi_a i$, i.e. $a \in \theta(j)^{t'}$ for some step $t' \geq t$. However, we know that $a \notin \mu^{NDAI}(j)$. Hence, the institution j is an interrupter for a, which contradicts the fact that there are no interrupters at x^*. Next, consider that $(a, h) \notin \mathcal{M}_i^t$. This case only happens if i is an interrupter over a and fulfill its quota, thus (a, h) is deleted from $\succ^{i,x}$ at some stage x of the NDAI algorithm. But this is not possible because the institutions did not fulfill its quota. This contradiction shows that $|\mu^{NDAI}(i)| = m_i$ for all $i \in I$.

Non-wastefulness. Follows from the fact that each institution fills its quota.

There is no justified envy. The proof follows the same steps as the proof in theorem 4.1. Let x^* be the last iteration of the NDAI. Consider, to the contrary, the existence of a pair (h, i) that has justified envy over a pair (h', i'), where $\tau(h) = i$ and $\tau(h') = i'$. Then, there exists an apartment a such that $\varphi^{NDAI}(h') = a$, $a \in \theta(i')$, and

i. $aP_h\varphi^{NDAI}(h)$,

ii. $(a, h) \in Ch_i(\mu^{NDA}(i) \cup \{(a, h)\}, m_i)$,

iii. $i\pi_a i'$.

Case 1. $i = i'$. The argument is the same as in Theorem 4.1.

Case 2: $i \neq i'$. The following Lemma is required to prove Case 2.

Lemma 5. *If i is an interrupter over an apartment a through a household h, then (h, i) has no justified envy over other pairs (h^a, i^a) at μ^{NDAI}, where $(a, h^a) \in \mu^{NDAI}(i^a)$.*

Proof of Lemma 5. Since i is an interrupter, there exist steps $\underline{t}$ and $\bar{t}$ such that $(a, h) \in \mu^t(i)$ for all $t \in [\underline{t}, \bar{t}]$, but $(a, h) \notin \mu^{t'}(i)$ for all $t' \geq \bar{t}$.

Because preferences of institutions are responsive, at $\bar{t}+1$ institution i drops (h, a) only if it has filled its quota and $\{(h^{\bar{t}+1}, a^{\bar{t}+1})\} \succ_i \{(h, a)\}$ for all $(h^{\bar{t}+1}, a^{\bar{t}+1}) \in \mu^{\bar{t}+1}(i)$.

Moreover, institutions improve along the sequence of tentative matchings, thus $(\overline{h}, \overline{a}) \succsim^i (\overline{h}^{\bar{t}+1}, a^{\bar{t}+1})$ for all $(\overline{h}, \overline{a}) \in \mu^{NDAI}(i)$ and $(\overline{h}^{\bar{t}+1}, a^{\bar{t}+1}) \in \mu^{\bar{t}+1}(i)$, and fills its quota at μ^{NDAI}, so $(h, a) \notin Ch_i(\mu^{NDAI}(i) \cup \{(h, a)\}, m_i)$, thus (h, i) has no justified envy over (h^a, i^a) at μ^{NDAI}, where $(a, h^a) \in \mu^{NDAI}(i^a)$. □

For the proof of Case 2, we now distinguish the following sub-cases.

Case 2.1 Consider that i is not an interrupter. Then i demands apartment a at iteration x^*. We also know that $a \in \theta^{NDAI}(i')$, so there exists a step t' such that $i' \pi_a j$ for all $j \in I_a^t$, for all $t \geq t'$. Since $a \notin \theta^{NDAI}(i)$ we have that $i' \pi_a i$, contradicting condition iii).

Case 2.1 If i is an interrupter, assuming that (h, i) has justified envy over (h', i'), through apartment a contradicts Lemma 5.

As a contradiction was obtained in all cases, this shows that there cannot be justified envy in the NDAI algorithm.

Pareto undomination. The proof follows the same steps as in Theorem 4.1. We show by induction that no household is rejected by an achievable apartment during the NDAI algorithm at iteration x^* where there are not interrupters.

Inductive hypothesis. At step t of the NDA algorithm, we assume that no household has been rejected by an achievable apartment. If a household is rejected by some apartment, this apartment is not achievable for her.

Inductive step. Assume that some household h^* is rejected at step $t+1$ from an apartment a. Consider by contradiction that a is achievable for household h^*. Thus, there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu(h^*) = (a, i^*)$. Let h be the household assigned to apartment a at the end of step $t+1$ and i be the institution that h belongs to..

Case 1. $i = i^*$. This case is similar to Theorem 4.1

Case 2. $i \neq i^*$. We know that $\varphi^\mu(h^*) = (a, i^*)$, i.e. the pair (a, h^*) is acceptable for the institution i^*, so $i^* \in I_a^{t+1}$. Since there are not interrupters at iteration x^*, the offer of i^* is rejected only if i fulfills its quota. So, there exists $(a_{\hat{x}}, h_{\hat{x}}) \in \mu^{t+1}(i^*)$ such that $\{(a_{\hat{x}}, h_{\hat{x}})\} \succ_{i^*} \{(a, h^*)\}$ where $\hat{x} = 1, 2, \ldots, m_{i^*}$. We know that $(a, h^*) \in \mu(i^*)$, i.e., some pair $(a_{\hat{x}}, h_{\hat{x}}) \notin \mu(i^*)$. By the inductive hypothesis, this implies that $h_{\hat{x}}$ prefers $a_{\hat{x}}$ to any other achievable apartment for her, i.e.

$$a_{\hat{x}} P_{h_{\hat{x}}} \varphi^\mu(h_{\hat{x}}). \tag{C.9}$$

Since $(a_{\hat{x}}, h_{\hat{x}}) \succ_{i^*} \varnothing$, by condition (C.9), we conclude that the pair $(h_{\hat{x}}, i^*)$ has justified envy over the pair (h^*, i^*) in the assignment μ, which contradicts the fact that μ is fair.

In all cases , a contradiction arises when we assume that household h^* is rejected by some achievable apartment a. So, no household is rejected by an achievable apartment. Therefore, μ^{NDA} is Pareto undominated by fair assignments.

Strategy-Proofness. Let P'_h be a misrepresentation of P_h resulting in the assignment of apartment a to h at institution i. We assume that P'_h is a successful misrepresentation of P_h; hence, we distinguish the following cases.

Case 1. Institution i is an interrupter for apartment a at some iteration x of the $NDAI[P]$. Then, there exists $(a', h') \in \mu^{NDA}[P](i)$ such that $(a', h') \notin \mu^{NDAI}[P'](i)$. Consider that $\varphi_A^{NDAI}[P'](h') = a''$, then

Case 1.1 $a' P_{h'} a''$, since i is an interrupter of a, (a, h) is rejected from i because $(a, h) \notin Ch_i(\mathcal{M}_i^{t'}, m_i)$ for all $t' \geq \bar{t}$, where $\bar{t}$ is the step where the interruption happened. Since $(a', h') \in \mu^{NDAI}[P](i)$ and institutions preferences are responsive, we have that

$$(a', h') \succ_i (a, h)$$

which implies that (h', i) has justified envy over (h, i) at $\mu^{NDAI}[P']$, which is a contradiction.

Case 1.2 $a'' P_{h'} a'$, then there exists household $h'' \neq h'$ such that $\varphi^{NDAI}[P](h'') = (a'', j)$. In this case $i = j$ or not.

Case 1.2.1 If $i = j$, then we have that $\varphi^{NDAI}[P](h'') P_{h''} \varphi^{NDAI}[P'](h'')$ or $\varphi^{NDAI}[P'](h'') P_{h''} \varphi^{NDAI}[P](h'')$. Note that the first case, is analogous to case 1.1, i.e., will end up generating a contradiction; in the last case, we can apply the same reasoning of h' on h'' that generates an infinite sequence of households $\{h^{(k)}\}$, which is not possible since H is finite. In any case a contradiction arises.

Case 1.2.2 If $i \neq j$, note that h' demands a'' during the $NDAI[P]$ algorithm but she does not get it despite the fact that (h', a'') is acceptable for i (remember that $(h', a'') \in \mu^{NDAI}[P'](i)$). Consequently, $j \pi_{a''} i$ because $(h'', a'') \in \mu^{NDAI}[P](j)$. However, $(h', a'') \in \mu^{NDAI}[P'](i)$, then there exists $a''' \neq a''$ such that $a''' \varphi_A^{NDAI}[P'](h'')$. Hence, we have that $a'''_{h''} a''$ or not. In the first case, (h'', j) has justified envy over (h', i) at apartment a'' in the assignment $\mu^{NDAI}[P']$ since $j \pi_{a''} i$, which is a contradiction. Otherwise, as in the case 1.2.2, we will end up generating an infinite sequence of households $\{h^{(k)}\}$.

In any case a contradiction arises. Thus, P' is not a successful misrepresentation of P when i is an interrupter of apartment a during $NDAI[P]$.

Case 2. Institution i is not an interrupter for apartment a. To prove that P'_h is not a successful misrepresentation, we first prove the following Lemma.

Lemma 6. *Consider a matching market where the over-demand condition holds. If an apartment is assigned at some step t in the NDA phase of the last iteration of the NDAI, this apartment is assigned under the assignment μ^{NDAI}.*

Proof of Lemma 6: Consider that x^* is the last iteration of the NDAI where there are no interrupters. Consider an apartment a that is assigned to some institution i at some step t. Since households iterate their demand to their match, $i \in I_a^{t+1}$. Hence this apartment is demanded to some institution at step $t + 1$. Since the over-demand condition holds, there are no interrupters and preferences are responsive, apartment a is assigned to some institution at the end of step $t+1$ (the institution in I_a^{t+1} with the highest priority at π_a). Iterating this argument, we conclude that apartment a is assigned to some institution at all steps $t' \geq t$. $\square$

Given that P' is a successful misrepresentation of P, we have that a is assigned to h or other households at some step t of the last iteration of the $NDAI[P]$. Hence, Lemma 6 guarantees that $\mu^{NDAI}[P](a) \neq \varnothing$, which allows us to adapt the reasoning of Theorem 4.1 to take into account distributional constraints and interrupters; remember that our goal is to get a contradiction from the fact that P' successfully misrepresents P when i is not an interrupter of a.

First, replacing P'_h with P''_h which puts a' at the top of h's preference list, we obtain:

Lemma 7. *Consider a matching market through institutions where the over-demand condition holds. Then for all h, $\varphi^{NDAI}[P'_h, P_{-h}](h) = \varphi^{NDAI}[P''_h, P_{-h}](h)$.*

Proof. Given that μ^{NDAI} is fair for households over the same institution when the over-demand condition holds, we apply the same reasoning as in Lemma 3.

□

Now, assume by contradiction the existence of household h^* and a successful misrepresentation P'_{h^*} of P_{h^*} such that $\varphi^{NDAI}[P](h^*) = a$ and $\varphi^{NDAI}[P'_{h^*}, P_{-h^*}] = a'$.

By Lemma 7, we can consider that P'_{h^*} is a preference list where a' is at the top of h's preference list. We prove by induction on the step at which h^* makes a match that no successful misrepresentation of preferences is possible.

Claim 4. Let x^* be the last iteration of the NDAI. Assume that h^* makes a match at t^*, with $1 \leq t^* \leq T$, then $\varphi^{NDAI}[P'](h) = \varphi^{NDAI}[P](h)$ for all h that makes a match at T. Moreover, if h^* makes a match at T, there is no profitable deviation P'_{h*} of her true preference list P_{h^*}.

Proof. By Lemma 7, the proof is the same as in Theorem 4.1. □

Now, we generalize Claim 3 for the NDAI case.

Claim 5. Suppose that h^* makes a match at t^*, $1 \leq t^* \leq T$ and P'_{h^*} and P_{h^*} are such that

$$\varphi^{NDAI}[P'_{h^*}, P_{-h^*}](h^*) R_{h^*} \varphi^{NDAI}[P](h^*).$$

Then $\varphi^{NDAI}[P'_{h^*}, P_{-h^*}](h^t) = \varphi^{NDAI}[P](h^t)$ for $h^t \neq h^*$ who makes a match at t, where $t^* \leq t \leq T$.

Proof. Note that Claim 4 generalizes Claim 1 for the NDAI case, that is the base of the proof of Claim 3. Hence, we follow the same reasoning as in Claim 3 to conclude that $\varphi^{NDAI}[P'_{h^*}, P_{-h^*}](h^t) = \varphi^{NDAI}[P](h^t)$ for $h^t \neq h^*$ who makes a match at t, where $t^* \leq t \leq T$. □

Now, we can continue with the reasoning of Theorem 4.1 to prove that the NDAI is strategy-proof. Thus, we consider that h^* makes a match at some step t of the $NDA[P]$ procedure, with $1 \leq t < T$. At any step where household h^* makes a match, Claim 5 implies that $\varphi^{NDAI}[P'](h^*) = \varphi^{NDAI}[P](h^*)$. Hence, there is no successful misrepresentation of P_h at an apartment a even when i is not an interrupter of it. Therefore, the NDAI is strategy-proof.

Participation criterion. Follows from Proposition 4.1.

Proof of Theorem 6.1:

Individual Rationality. Follows from the same argument as in theorem 4.1.

Distributional Constraints(feasibility). The proof follows the same steps as in Theorem 5.1.

Non-wastefulness. Follows from the fact that each institution fills its quota.

There is no justified envy over households that belong to the same institution. Let x^* be the last iteration of the NDAI. By contradiction, we consider the existence of a pair (h, i) that has justified envy over a pair (h', i), where $i \in \tau(h) \cap \tau(h')$. Then, there exists an apartment a such that $\varphi_A^{NDA}(h') = a$, $a \in \theta(i)$, and

i. $aP_h\varphi_A^{NDA}(h)$,

ii. $(a, h) \in Ch_i(\mu^{NDA}(i) \cup \{(a, h)\}, m_i)$.

The proof in Case 1 is identical to the proof of Theorem 5.1.

Note that Case 2 can not be extended when households are attached to multiple institutions because household h can form a blocking pair with another institution $i' \neq i$, as in example 6.1.

Pareto undomination. We proceed by induction to show that no household is rejected by an achievable apartment during the NDA algorithm at iteration x^* where there are no interrupters.

Inductive hypothesis. At step t of the NDA algorithm, we assume that no household has been rejected by an achievable apartment. If a household is rejected by some apartment, this apartment is not achievable for her.

Inductive step. Consider that some household h^* is rejected at step $t + 1$ from an apartment a. By contradiction, we consider that a is achievable for household h^*. Thus, there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu(h^*) = (a, i^*)$. So, the pair (a, h^*) is acceptable for the institution i^*.

Next, let h be the household assigned to the apartment a at the end of step $t + 1$, this means that $\varphi^{t+1}(h) = (a, i)$ where $i = \tau(h)$. We analyze the following cases.

Case 1. $i \in \tau(h^*)$. Since $\varphi^{t+1}(h) = (a, i)$, apartment a belongs to $\theta^{t+1}(i)$. Hence

$$\{(a, h)\} \succ_i \{(a, h^*)\} \tag{C.10}$$

because $(a, h^*) \notin \mu^{t+1}(i)$. Since preferences are responsive, $(a, h) \in Ch_i(\mu(i) \cup \{(a, h)\}, m_i)$. Note that h prefers a to all the apartments that have not rejected her, so that the induction hypothesis ensures that household h prefers a to any other achievable apartment for her, $aP_h\varphi_A^\mu(h)$. Moreover, $(a, h^*) \in \mu(i)$. Hence the pair (h, i) has justified envy over the pair (h^*, i) at apartment a in assignment μ, which contradicts the fact that μ is a fair assignment.

Case 2. $i \notin \tau(h^*)$. We know that $\varphi^\mu(h^*) = (a, i^*)$, i.e. the pair (a, h^*) is acceptable for the institution i^*, so $i^* \in I_a^{t+1}$. Since there are no interrupters at iteration x^*, the offer of i^* is rejected only if i fulfills its quota. So, there exist $(a_{\hat{x}}, h_{\hat{x}}) \in \mu^{t+1}(i^*)$ such that $\{(a_{\hat{x}}, h_{\hat{x}})\} \succ^{i^*} \{(a, h^*)\}$ where $\hat{x} = 1, 2, \ldots, m_{i^*}$. We know that $(a, h^*) \in \mu(i^*)$, i.e., some pair $(a_{\hat{x}}, h_{\hat{x}}) \notin \mu(i^*)$. By the inductive hypothesis, this implies that $h_{\hat{x}}$ prefers $a_{\hat{x}}$ to any other achievable apartment for her, i.e.

$$a_{\hat{x}} P_{h_{\hat{x}}} \varphi_A^\mu(h_{\hat{x}}). \tag{C.11}$$

Since $(a_{\hat{x}}, h_{\hat{x}}) \succ^{i^*} \varnothing$, by (C.11), we conclude that the pair $(h_{\hat{x}}, i^*)$ has justified envy over the pair (h^*, i^*) in the assignment μ, which contradicts the fact that μ is fair.

In all cases, a contradiction arises when we assume that household h^* is rejected by some achievable apartment a. So, no household is rejected by an achievable apartment. Therefore, μ^{NDA} is Pareto undominated by fair assignments.

Strategy-proofness. The prove is analogous to the proof of Theorem 5.1. In the following Lemma we generalize Case 1 in Theorem 5.1 when households are attached to multiple institutions.

Claim 6. Consider a market through institutions with distributional constraints where the over-demand condition holds. Consider a pair (a, h) such that $aP_h\varphi_A^{NDAI}[P](h)$. There is no misrepresentation P'_h of P_h such that $\varphi_A^{NDAI}[P'_h, P_{-h}](h) = a$.

Proof. Let P'_h be a misrepresentation of P_h where apartment a is acceptable. We distinguish the following cases.

Case 1. All institutions in $i \in \tau(h)$ are interrupters for the apartment a. As a consequence, all pairs (a, h) are deleted from the priority $\succ_{i,x_i}$, i.e., pairs (a, h) are not acceptable at priority $\succ^{i,x'}$ for all $x' > x_i$. So, household h is not assigned to apartment a in stages $x'.1$, with $x' > x_i$. Therefore, P'_h is not a successful misrepresentation of P_h.

Case 2. There exists an institution $i \in \tau(h)$ such that i is not an interrupter for apartment a. We proceed as in Theorem 5.1 to prove that no misrepresentation of P_h is successful. □

Claim 6 is used in addition to the arguments of Theorem 5.1 to show that the NDAI algorithm is strategy-proof.

Participation criterion. Follows from Proposition 4.1.

□

Appendix D. Comparison with Kamada and Kojima [4]

In this appendix we show the connection between the market with distributional constraints and regional preferences [4] and our model. A market with distributional constraints and regional preferences $\tilde{E} = (D, H, Q, R, \tau, P, \succ, \succsim, \tilde{Q})$ is defined by:

1. $D = \{d_1, d_2, \ldots, d_D\}$ is a finite set of doctors, a generic doctor is denoted by d;
2. $H = \{h_1, h_2, \ldots, h_H\}$ is a finite set of hospitals, a generic hospital is denoted by h;
3. $Q = (q_{h_1}, q_{h_2}, \ldots, q_{h_H})$ is a vectors of quotas, where q_h is the quota of the hospital h, a generic quota is q;
4. $R = \{1, 2, \ldots, R\}$ is a finite set of regions, a generic region is r;
5. $\tau : H \to R$ is the region function, i.e. if a hospital h belongs to the region r, we write that $\tau(h) = r$. Let H_r be the set of hospitals in region r, note that $H_r \cap H_{r'} = \varnothing$ for region $r' \neq r$;
6. $P = (P_{d_1}, P_{d_2}, \ldots, P_{d_D})$ is the vector of doctors' preferences, P_d is the strict preference of household $h \in H$ over $H \cup \varnothing$; hP_dh' means that doctor d prefers h to h', a hospital h is acceptable for doctor d if $hP_h\varnothing$.

7. $\succ = (\succ_{h_1}, \succ_{h_2}, \ldots, \succ_{h_H})$ is the profile of hospitals priorities over the set of doctors D. We assume that for each $h \in H$ the preference $\succ_h$ is responsive on 2^D, i.e. for any $d, d' \in D$ and $S \in 2^D$ we have that

   i. $S \cup \{d\} \succ_h S \cup \{d'\}$ if and only if $d \succ_h d'$, and

   ii. $S \cup \{d\} \succ_h S$ if and only if $d \succ_h \varnothing$;

8. $\succsim_r$ is the regional preference of r over the set of vectors $W_r = \{w = (w_h)_{h \in H} | w_h \in \mathbb{Z}_+\}$, where w_h specifies the number of doctors allocated to each hospital h in region r;

9. There exists a vector of regional caps $\tilde{Q} = (q_r)_{r \in R}$, where q_r is a non-negative integer for each region r.

Kamada and Kojima [4] introduce quasi-choice rules which pick the preferred capacity vector given the regional cap. Given $\succsim_r$, a function $\tilde{Ch}_r : W_r \times \mathbb{Z}_+ \to W_r$ is an **associated quasi choice rule** if $\tilde{Ch}_r(W_r, q_r) \in argmax_{\succeq_r}\{w \in W_r || w| \leq q_r\}$ for any non-negative $w = (w_h)_{h \in H_r}$. They also require that the quasi choice rule $\tilde{Ch}_r$ be **consistent**, that is, $\tilde{Ch}_r(\omega) \leq \omega' \leq \omega \Rightarrow \tilde{Ch}_r(\omega') = \tilde{Ch}_r(\omega)$. In other words, if $\tilde{Ch}_r$ is still available when the capacity vector reduces to $\omega' \leq \omega$, then the associated quasi-choice rule chooses $\tilde{Ch}_r(\omega')$. They also assume that the regional preferences $\succeq_r$ satisfy the following regularity conditions:

(1) $\omega' \succ_r \omega$ if $\omega_h > q_h \geq \omega'_h$ for some $h \in H_r$ and $\omega'_{h'} = \omega_{h'}$ for all $h' \neq h$. In words, no hospital wants more doctor than its real capacity. This implies that $[\tilde{Ch}_r(w)]_h \leq q_h$ for each $h \in H_r$.

(2) $\omega' \succ_r \omega$ if $\sum_{h \in H_r} \omega_h > q_r \geq \sum_{h \in H_r} \omega'_h$. So, each region prefers the total number of doctors in the region to be at most its regional cap.

(3) If $\omega' \lneq \omega \leq q_{H_r} := (q_h)_{h \in H_r}$ and $\sum_{h \in H_r} \omega_h \leq q_r$, then $\omega \succ_r \omega'$. In other words, each region prefers to fill as many positions of hospitals in the region while the regional cap would not be violated.

Regional preferences $\succsim_r$ are said to be **substitutable** if there exists an associated quasi choice rule $\tilde{Ch}_r$ that satisfies $w \leq w' \Rightarrow \tilde{Ch}_r(w) \geq \tilde{Ch}_r(w') \wedge w$.

Next, Kamada and Kojima [4] define stable matchings in markets with distributional constraints and regional preferences:

A **matching** μ is a function that satisfies

(i) $\mu(d) \in H \cup \{\varnothing\}$ for all $d \in D$,

(ii) $\mu(h) \subseteq D$ for all $h \in H$ and

(iii) for any $d \in D$ and $h \in H$, $\mu(d) = h$ if and only if $d \in \mu(h)$.

A matching is **feasible** if $\mu(r) \leq q_r$ for all $r \in R$, where $\mu_r = \bigcup_{h \in H_r} \mu(h)$.

A matching μ is **stable** if it is feasible, individually rational, and if (d, h) is a blocking pair, then

(i) $|\mu(h)| = q_{r_h}$,

(ii) $d' \succ_h d$ for all doctors $d' \in \mu(h)$, and

(iii) either $\mu(d) \notin H_{r(h)}$ or $w \succsim_{r(h)} w'$,

where $w_{h'} = |\mu(h')|$ for all $h' \in H_{r(h)}$ and $w'_h = w_h + 1$, $w'_{\mu(d)} = w_{\mu(d)} - 1$ and $w'_{h'} = w_{h'}$ for all other $h' \in H_{r(h)}$.

In order to find an assignment between hospitals and doctors that respect the distributional constraints and regional caps, Kamada and Kojima introduced the Generalized Flexible Deferred Acceptance (GFDA) algorithm.

For each region r fix a quasi-choice rule $\tilde{Ch}_r$. The GFDA algorithm proceed as follows

1. Begin with an empty matching, i.e. $\mu_d = \varnothing$ for all $d \in D$.
2. Choose a doctor d arbitrarily who is currently not tentatively matched to any hospital and who has not applied to all acceptable hospitals yet. If such a doctor does not exist, then terminate the algorithm.
3. Let d apply to the most preferred hospital $\overline{h}$ at H_d among the hospitals that have not rejected d so far. If d is unacceptable to $\overline{h}$, then reject this doctor and go back to step 2. Otherwise, let r be the region such that $\overline{h} \in H_r$ and define vector $\omega = (\omega_h)_h \in H_r$ by

   (a) $\omega_{\overline{h}}$ is the number of doctors currently held at $\overline{h}$ plus one, and

   (b) w_h is the number of doctors currently held at h if $h \neq \overline{h}$,
4. Each hospital $h \in H_r$ considers the new applicant d (if $h = \overline{h}$) and doctors who are temporarily held from the previous step together. It holds its $(\tilde{Ch}_r(w))_h$ most preferred applicants among them temporarily and rejects the rest (so doctors held at this step may be rejected in later steps). Go back to step 2.

We adapt the NDA algorithm to the model of Kamada and Kojima [4]. The algorithm differs from the NDA algorithm of our baseline model in two respects: (i) unmatched doctors are selected sequentially rather than simultaneously to make offers and (ii) we replace the choice function of regions by the quasi-choice function of regions. Formally:

Initialization

Consider a market $(D, H, Q, R, \tau, P, \succ, \succsim, \tilde{Q})$ with distributional constraints and regional preferences. The matching is initialized to be the empty matching, so $\mu^0(h) = \mu^0(d) = \mu^0(r) = \varnothing$, for all $d \in D$, $h \in H$ and $r \in R$.

For all doctors $d \in D$, let $H^t_d := H$, and $t = 1$. For each region r, fix a quasi-choice rule $\tilde{Ch}_r$.

$\mathbf{A}^t$. Eliciting the demand of doctors

Arbitrarily pick one unassigned doctor d, who asks for the most preferred hospital in H^t_d, denoted by D^t_d while r is the region of D^t_d; moreover matched doctors d' in region r iterate their demand to their match, $D^t_{d'} = \{\mu^{t-1}(d)\}$.

For all hospitals $h \in H$ in region r, we define the set of doctors that demand hospital h in region r as follows:

$$D^t_{h,r} = \{d \in D | D^t_d = \{h\}, d \succ_h \varnothing \text{ and } \tau(h) = r\}.$$

The set of pairs (d, h) that can be assigned to region r is defined as

$$M^t_r = \{(d, h) \in D \times H | d \succ_h \varnothing \text{ and } d \in D^t_{h,r}\}.$$

The possible assignments are

$$\mathcal{P}^t_r = \left\{p = \{(d, h)\}_{\tau(h)=r} \mid (d, h) \in M^t_r \text{ and } d \text{ is not matched twice}\right\}.$$

The number of doctors matched to hospital h at p is $w_h(p) = |\{d \in D \mid (d, h) \in p\}|$, thus, the set of capacity vectors is

$$W^t_r = \left\{w = (w_h)_{h \in H} | \exists\ p \in \mathcal{P}^t_r \text{ and } w_h = w_h(p) \text{ for all } h \text{ in region } r\right\}.$$

B^t. Matching the demand of region r and hospitals of the region.

B^t.1 Regions r demands the vector

$$\omega^t_r = (\omega^t_h)_{\tau(h)=r} = \tilde{Ch}_r(W^t_r, q_r).$$

B^t.2 Each hospital h in region r is tentatively assigned to the preferred subset of $D^t_{h,r}$ with cardinality w^t_h. The assignment in other regions remains the same.

C^t. Iteration over $\mathbf{D^t_d}$

Let $H^{t+1}_d := H^t_d \setminus \{\max_{P_d} H^t_d\}$, $t := t + 1$.

If all doctors have been rejected by all the apartments in her preference list or is matched, the tentative assignment becomes the outcome assignment. Otherwise, go to the Phase A^{t+1}.

The assignment produced by the previous algorithm is denoted by $\tilde{\mu}^{NDA}$. It depends on a market $\tilde{E}$ and a fixed associated quasi choice rule $\tilde{Ch}$. We now state that the NDA produces a stable and strategy-proof matching in the market with distributional constraints and regional preferences:

Theorem Appendix D.1. *Suppose that regional preferences $\succeq_r$ are substitutable for all $r \in R$. Then the matching produced by the nested deferred acceptance algorithm is stable and strategy proof for doctors.*

Proof. We adapt the proof of Kamada and Kojima [4], based on matching with contracts, to the Nested Deferred Acceptance algorithm with regional preferences.

First, as they do, we establish the relation between matching markets with regional preferences and matching with contracts. So, let $X = D \times H$ be the set of contracts. Note that, for each doctor d, the preference profile P_h induces a preference relation $\tilde{P}_h$ over $(\{d\} \times H) \cup \{\varnothing\}$ in the following way $(d, h')\tilde{P}_d(d, h)$ if and only if $h' P_d h$. Moreover, we say that $(d, h)\tilde{P}_d\varnothing$ if hospital H is unacceptable under P_d.

Now, for each region $r \in R$, we define preferences $\succeq_r$ and its associated choice rule $\overline{Ch}_r$ over all subsets of $D \times H_r$. For any $X' \subset D \times H_r$, let $\omega(X') := (w_h(X'))_{h \in H_r}$ be the vector such that $w_h(X') = |\{(d,h) \in X' | d \succ_h \varnothing\}|$. For each X', the chosen set of contracts $\overline{Ch}_r(X')$ is defined by

$$\overline{Ch}_r(X') = \bigcup_{h \in H_r} \left\{ (d,h) \in X' \middle| |\{d' \in D | (d',h) \in X', d' \succeq_h d\}| \leq (\tilde{Ch}_r(\omega(X')))_h \right\}. \tag{D.1}$$

That is, each hospital $h \in H_r$ chooses its $(\tilde{Ch}_r(\omega(X')))_h$ most preferred contracts available in X'. The domain of the choice rule $\overline{Ch}_r$ can be extended to all subsets of X by

$$\overline{Ch}_r(X') = \overline{Ch}_r(\{(d,h) \in X' | h \in H_r\})$$

for any $X' \subseteq X$.

Definition 2. Hatfield and Milgrom [33]. Choice rule $\overline{Ch}_r(\cdot)$ satisfies the **substitutes condition** if there does not exist contracts $x, x' \in X$ and a set of contracts $X' \subseteq X$ such that $x' \notin \overline{Ch}_r(X' \cup \{x'\})$ and $x' \in \overline{Ch}_r(X' \cup \{x, x'\})$.

Definition 3. Hatfield and Milgrom [33]. Choice rule $\overline{Chr}(\cdot)$ satisfies the **law of aggregate demand** if for all $x' \subseteq X'' \subseteq X$, $|\overline{Ch}_r(X')| \leq |\overline{Ch}_r(X'')|$.

Proposition Appendix D.1. *Suppose that $\succeq_r$ satisfies the substitutes condition. Then the choice rule $\overline{Ch}_r(\cdot)$ defined above satisfies the substitutes condition and the law of aggregate demand.*

Proof. This proposition follows from Proposition 1 of Kamada and Kojima [4]. □

A subset X' of $X = D \times H$ is said to be **individually rational** if (1) for any $d \in D$, $|\{(d,h) \in X' | h \in H\}| \leq 1$, and if $(d,h) \in X'$ then $hP_d\varnothing$, and (2) for any $r \in R$, $\overline{Ch}_r(X') = X' \cap (D \times H_r)$.

Definition 4. A set of contracts $X' \subseteq X$ is a **stable allocation** if

(1) it is individually rational, and

(2) there exists no region $r \in R$, hospital $h \in H_r$, and a doctor $d \in D$ such that $(d,h)\tilde{P}_d x$ and $(d,h) \in \overline{Ch}_r(X' \cup \{(d,h)\})$, where x is the contract that d receives at X' if any and $\varnothing$ otherwise.

When condition (2) is violated by some (d,h), we say that (d,h) is a **block** of X'. A doctor-optimal stable allocation in the matching model with contracts is a stable allocation that every doctor weakly prefers to every other stable allocation [33].

Given any individually rational set of contracts X', define a corresponding matching $\mu(X')$ in the original model by setting $\mu(d)(X') = h$ if and only if $(d,h) \in X'$; and $\mu(d)(X') = \varnothing$ if and only if no contract associated with d is in X'. Since each doctor regards any set of contracts with cardinality of at least two as unacceptable, each doctor receives at most one contract at X' and hence $\mu(X')$ is well defined for any individually rational X'.

Proposition Appendix D.2. *If X' is a stable allocation in the associated model with contracts, then the corresponding matching $\mu(X')$ is a stable matching in the original model.*

Proof. Follows from Proposition 2 of Kamada and Kojima [4]. □

Remark 2. We recall the connection between the Nested Deferred Acceptance and the Cumulative Offer Process of Hatfield and Milgrom [33]. If doctor d asks for her most preferred hospital h at some step in the NDA, then contract (d, h) is proposed at the same type of the cumulative offer process. Also, the set of doctors accepted by a hospital at some step of the NDA is equivalent to the set of contracts held at the corresponding step of the cumulative offer process. Thus, if X' is the output of the cumulative offer process, then $\mu(X')$ is the matching generated by the NDA.

We are now ready to conclude the proof of Theorem 6.1. By Proposition 1, the choice function of each region satisfies the substitutes condition and the law of aggregate demand in the associate model of matching with contracts. By Hatfield and Milgrom [33], Hatfield and Kojima [34], and Hatfield and Kominers [35], the cumulative offer process with choice functions satisfying these conditions produces a stable allocation and is strategy-proof. The former fact, together with Remark 2 and Proposition Appendix D.2, implies that the outcome of the Nested Deferred Acceptance Algorithm is a stable matching in the original model. By Remark 2, we conclude that the NDA mechanism is strategy-proof for doctors. □