# Matching through Institutions

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**Abstract**

We analyze a matching market where institutions pool their objects so as to enlarge the choice set of their agents. They set a common assignment taking into account both the preference of agents over objects and the preferences of institutions over agent-object pairs. Furthermore institutions keep control over their objects by setting which of the other institutions have priority over their objects. When there are no distributional constraints, we show the existence of a fair assignment by introducing the Nested Deferred Acceptance (NDA) algorithm, which reaches a Pareto optimal assignment among fair matchings. This procedure nests a one-to-one matching between agents and objects and a one-to-many matching between objects and institutions, it is strategy-proof for households. If all objects are overdemanded, existence of a fair assignment is also guaranteed under distributional constraints, although one needs to sophisticate the mechanism to get rid of interruptions. Importantly, priorities of institutions over agent-object pairs can follow a perfect complement pattern. Finally we allow agents to belong to many institutions, which restrict the previous results to fairness over agents of the same type.

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## 1. Introduction

This paper studies matching markets intermediated by institutions. Institutions own objects and agents are attached to institutions. In autarky, institutions assign their objects to their agents. However, they may also consider a more flexible assignment by pooling objects with other institutions, thereby expanding the choice set of all agents. In this paper, our objective is to study these flexible assignment rules where: a) each object has a priority list over institutions that indicates to which institutions the owner institution pools the object. If all institutions are acceptable according to the priority order, the object is pooled with all institutions; on the opposite, the object is not pooled when its owner is the only acceptable institution in its priority list (all intermediate cases are allowed); b) agents have preferences over the entire set of objects, and c) institutions have priorities over the assignment of objects to its agents. Thus, matching through institutions is a three-sided matching market. We further consider the

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case that institutions face distributional constraints: the number of agents attached to one institution who receive an assignment must be equal to a fixed quota, typically equal to the number of objects that the institution owns.

Matching through institutions occurs in a large variety of situations. Our leading example is social housing in Paris, where four types of institutions (the French state, the city of Paris, local councils and private firms) owns the residential real estate dedicated to social housing. While the property rights over buildings are exclusive, the rent of apartments is open/accessible for households belonging to any institution assigned by a centralized procedure with the following features. First, each institution receives a fixed quota of apartments corresponding to the number of vacant apartments they bring to the pool. Then households report the apartments in the pool they would like to hire. Importantly institutions care about the assignment of apartments to households, either on the basis of the preferences of their applicants (this is typically the case for the local councils and private firms) or to maintain distributional objectives and respect a balance between different types of applicants (this is typically the case for the French state and the city of Paris, given priority of large apartments to large families). The final assignment is set by committee involving all institutions, it consist in allocating apartments to institutions, taking into account their quota.[1] We observe that when an apartment is temporarily rent by a household, of any institution, it does not affect its property right.

The assignment of seats in study-abroad programs in American colleges and universities is another example of matching through institutions. Many liberal arts colleges maintain study-abroad programs with permanent staff and office space. Because all colleges cannot be present in all countries, they pool resources, allowing students from other colleges and universities to participate in their programs. These agreements expand the choice set of students, by increasing the number of countries where they can spend terms abroad. These agreements are sometimes based on transfer payment and sometimes made on a quid pro quo basis, where the number of incoming and outgoing students are matched for each university.[2] In this assignment problem, students have preferences over seats. Colleges fix a priority over other colleges for every seat in a study- abroad program, typically giving absolute priority to college that spend a large amount of resources to fund seats. Finally, universities have priorities over assignments of students to study abroad, either to increase the welfare or their students or to respect distributional constraints, allowing for sufficient diversity in the study-abroad programs.

Expanding the choice set of students is also the objective of inter-district school choice programs, like the one sponsored by the state of New Jersey.[3] These programs allow school districts to exchange students and to specialize in specific programs, like arts programs or programs for students with special needs. Pupils from one school district can be assigned to a school in another district up to a limit in

[1]For a detailed description of the mechanism, see the annual report on the assignment of social housing in Paris [1].

[2]In most cases, the agreements among colleges are bilateral. In some cases, like in the University of California system, there exists a central clearinghouse assigning students to the programs offered by different universities.

[3]See www.nj.gov/education/choice for a description of the system.

the number of seats. Inter-district school programs are either based on financial transfers across school districts or, as in the case of New Jersey, on state funding, which sets a priority for each seat over school districts.

A final example of matching through institutions are tuition exchange programs for children of staff and faculty of universities.[4] Start from a situation where children of staff are offered free or reduced tuition only in their own universities. Through tuition exchange programs, universities pool their resources so that children of staff can benefit from reduced tuition in a larger set of universities. In order to maintain financial balance, the number of incoming and outgoing students in each university must be equalized. Both in inter-district school choice and tuition exchange programs, students have access to a larger set of seats. Seats may be assigned in a preferential way to some institutions (pupils from the local school district, children of staff from the university). School districts and universities either care only about the welfare of their own pupils or children of staff or have more general objectives over the assignment of seats to their students.

In contrast with the vast literature on mechanism design, which mainly studies two-sided matching markets with exclusive use of objects by owners, we model matching through institutions as a three-sided market involving agents (which we refer to as "households"), objects (which we refer to as "apartments"), and owners (which we refer to as "institutions") that potentially can use objects from different owners. Our objective is to obtain assignments that satisfy the classical properties of individual rationality, non-wastefulness, fairness, and strategy-proofness, adapted to this new setting where apartments are a common pool of resources. For dealing with three preference/priority orders, classical mechanisms, including the cumulative offer mechanism [3] for matching with contracts, are generally not operative for matching through institutions. The main contribution of the paper is to propose a new algorithm, the Nested Deferred Acceptance (NDA), which nests two different deferred acceptance algorithms that interact in parallel to deal with the flexibility of using objects. In the first one (the "outer loop"), each household asks for her most preferred apartment among those that have not yet been denied to her. Given this list of demands, we run a second deferred acceptance algorithm (the "inner loop") among institutions. Each institution chooses a set of household-apartment pairs that maximizes its priority and does not exceed its vector of quotas. If more than one institution is interested in one apartment, it is assigned according to the priority of each apartment. Hence, the inner loop may assign apartments to institutions that do not initially own them. Thus, we observe that institutions do not demand apartments (paired with households) in the decreasing order of their priorities, rather the demand is contingent on the demand expressed by households. The process is repeated until apartments are assigned to households in such a way that institutions respect their quotas. Going back to the outer loop, we next ask rejected households to apply for their next preferred apartment, and the procedure continues until no household is rejected. If institutions do not share apartments, this algorithm is equal to the Students Optimal Stable Mechanism

[4]See Dur and Uñver [2] for a description of these programs.

(SOSM).

When there is no distributional constraints, the classical theory on two-sided matching naturally extends to our three-sided problem, in particular, we prove that the NDA mechanism is strategy-proof for households and outputs a fair assignment that is Pareto undominated by any fair assignment. We are not the only ones who nest mechanisms. Delacretaz, Kominers, and Teytelboym [4] do something similar in their Maximum Rank Deferred Acceptance (MRDA) algorithm, in which they nest the interaction between localities (institutions) and families (households) to update a maximum rank; this nested step generates a rejection chain under families truthfully report their preferences to the mechanism. Despite that the MRDA and NDA algorithms are based on the DA, these mechanisms are different in two key aspects. First, the MRDA does not deal with the possibility to share housing among localities since they have exclusive use over housing that they own. Second, the NDA is equivalent to the SOSM when there is exclusive use of apartments, which is not stable in the refugee resettlement problem given the multidimensional services features and constraints.

In contrast, imposing distributional constraints, specifically fixed quotas to be fulfilled by the different institutions, involves two major complications: first, it may lead to the nonexistence of feasible assignments; second, even when feasible assignments exist, they may be subject to justified envy. In order to guarantee existence of feasible assignments, we assume that each institution/agent has a sufficiently long list of acceptable agents/apartments so that every apartment is acceptable to some agent in the list. Under this sufficient condition –termed the over-demand condition, which is natural since we impose quotas to be fulfilled– we show that assignments satisfying distributional constraints exist. Distributional constraints hurt the NDA's well-behavior; when they are effective, this mechanism may not generate a fair assignment. We observe that justified envy arises because of the presence of interruptions caused by institutions. To restore fairness, we follow a constructive argument, which involves a modified version of the NDA algorithm that now identifies and deletes "interrupters": they are institutions which are temporarily assigned to apartments that they will not be assigned to in the final matching for receiving better demands from their agents, thereby blocking tentatively access of an apartment to households from other institutions, demands that have to be iterated once the apartment is dropped by the first institution so as to reach a fair assignment. The problem arises because institutions do not demand household-apartments pairs in the decreasing order of their priorities, rather their demand is contingent to the demand expressed by households. The phenomenon is thus different from the "interrupters" observed in Kesten [5], where interrupters never drop their tentative demand and have to be deleted to obtain Pareto efficiency, not fairness. The solution to both phenomena, however, is similar since, as in Kesten [5], we solve the problem by modifying the priority of each interrupters Specifically, the NDA with Interrupters (NDAI) improves the NDA by dropping apartments from the priority of interrupter institutions. Our main result shows that the output given by the NDAI is fair, Pareto undominated by any other fair assignment satisfying the distributional constraints and strategy-proof. Recall that, by contrast, the Efficiency Adjusted Deferred Acceptance Mechanism (EADAM) of Kesten is manipulable because interrupters are students who strategically report their preferences while in our case households are not interrupters. We also analyze the case where the same agent can appear on the list of multiple institutions (This may be the case in the assignment of social housing in Paris). We show that fair assignments do not necessarily exist and that the NDAI then produces a matching which is not fair; as it is expected in this more general three-sided matching formulation, but only fair among agents belonging to the same institution. Finally, we observe that institutions guarantee their households a Pareto improvement with respect to the non-pooling case by entitling top priority to themselves, on their goods. In this sense, the mechanism is individually rational and satisfies the participation constraint.

When institutions do not face quota restriction, the inner loop of the NDA algorithm does not iterate. Indeed institutions only act on the receiving side of the market, where their acceptance is concatenated with one of the apartments. In this case, one might model matching through institution as the matching with contracts model and adapt the cumulative offer mechanism. This is not true under quota restrictions where institutions receive offers from households and only make offers to apartments that belong to the favorite household-apartment set of pairs that fulfills its quota. Typically, in a DA algorithm, offers are accepted neither in the decreasing nor the increasing order of the priorities of the receivers without generating interrupters. We can see that institutions are also in the offering side of the market by remembering that, in the NDAI, institutions do not do make offers in the decreasing order of their priorities, which generates the presence of interrupters. We observe, however, that the NDA is a versatile tool that can be useful also in two-sided markets where demands and other criteria can be concatenated, in particular, maximum quotas. While we deal with strict quotas in the main text, we show that it also applies to settings with maximum quotas, as in Kamada and Kojima [6]. Inspired by the assignment of doctors to hospitals with flexible regional quotas, they propose a very general model where matching can be flexible across hospitals in the same region, and regions have preferences over the distribution of doctors to hospitals in their jurisdictions. In an Appendix, we show that the NDA algorithm can be adapted to matching with distributional constraints, by reinterpreting regions as "institutions," doctors as "households" and jobs in hospitals as "apartments." The NDA, however, does not adapt to general distributional constraints, like Goto et al. [7].

### *1.1. Relation with the literature*

**Three-sided markets**. The paper is related to the literature of three-sided markets. This literature focuses on analyzing the difficulties to find a stable, whenever it exists, on different preferences domains: cyclic preferences (Biró and McDermid [8]), binary relations (Farczadi, Georgiu and Köneman [9]), and hybrid preferences (Zhang and et al. [10]). Previous works indicate that the existence of a stable matching is an NP-problem, and emphasize on the possibility to define different notions of stability in such markets (Alkan [11] and Huang [12]). We contribute to this literature by providing a setting where it is possible to find a stable (fair) matching by using the NDA, which is an algorithm that runs in polynomial time. In our setting, priorities of institutions are defined over the other two sides of the market, while households care about apartments and apartments prioritize institutions. Unlike Biró and Mc Dermid [8], there exist a fair assignment even in the presence of exact quotas (distributional constraints) by assuming that households only belong to one institution. When households are attached to multiple institutions, we extend the previous result by imposing the over-demand condition. Although such condition does not guarantee the existence of fair assignments on such markets, the NDAI finds fair over households of the same type assignments whenever the condition holds.

**Perfect complements.** The matching with contracts approach (Hatfield and Milgrom [3]) has been successful in relaxing the substitute condition by Kelso and Crawford [13] and establishing that the Generalized Gale-Shapley algorithm outputs a stable allocation, even in settings where the set of stable matchings has no lattice structure, like in Hatfield and Kojima [14], Sun and Yang [15], Sönmez and Switzer [16], Aygün and Sönmez [17], Kominers and Sönmez [18], and Hatfield and Kominers [19]. Hatfield, Kominers, Nichifor, Ostrovsky and Westkamp [20] propose a model that cap the notions of substitutability that appears in different economic environments and relate them. Contracts, however, are two-sided, which limits the scope of the analysis to pairwise stability, which is not the natural stability restriction in our application where the use of objects can be shared among all institutions.

Our problem can also be modeled as a network, as in Ostrovsky [21] and Hatfield and Kominers [22], who extends the analysis and fixed point techniques to supply chain networks and establishes the stability of upstream-downstream contracts under same-side substitutability and a new substitutable condition, called cross-side complementary. Our three-sided approach is less general than Ostrovsky [21] since we do not deal with networks of any size, and our preference/priority structure is also restricted. In contrast, it is less restrictive in term of complementarity. Rephrasing our problem in his setting, specifically section V.C, two-sided markets with complementarities, one side of the market can be interpreted as institutions that see both types of agents on the other side, households, and apartments, as complements. While Ostrovsky [21] and Hatfield and Kominers [22] require households and apartments to be substitutable for one another, we dispense of this assumption. Another difference lies in that fact that households have preferences defined on apartments, not on institutions.

Most of the previous literature studies many-to-one problems when establishing the different ways agents are substitutable to one-another. This aspect is present in our model, specifically when an institution matches various households-apartment pairs, we simply assume that priorities orders on these pairs comply with the property of responsiveness, thus in this respect, we are less general than the previous models, especially Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp [20].

The complementarity we deal with is of another type, critically it is already presented when an institution matches only with one household-apartment pair: the institution is allowed to rank as first option a given large family with a three bedrooms apartment and as a second option a small family with a two-bedroom apartment. Thus, if the institution is not assigned the first apartment, it drops the large family and demands the second apartment for the small family.

Matching with couples is another branch of the literature considering preferences/priorities exhibiting complementarities. Typically, stability is not met but under restrictions on preferences/priorities (Blum, Roth and Rothblum [23], Cantala [24], Klaus and Klijn [25], Pycia [26], Sethuraman et al. [27] and Nguyen and Vohra [28]), or with high probability in large markets (Kojima, Pathak and Roth [29], Ashlagi et al. [30]).

Finally, in many-to-one markets with complements, there is no fair assignment when one observes the phenomenon of a domino effect: a firm drops a worker because her complement is leaving the firm. Hatfield and Kominers [31] study similar dynamics when the cardinality of objects to be assigned varies in size.

**Matching with constraints**. The problem we consider is related to the rapidly emerging literature on matchings with constraints. In the school choice context, a number of recent papers have analyzed the effect of constraints resulting from affirmative action considerations. One stream of papers interpret affirmative action as "leveling the playing field," as in Kojima [32], Hafalir, Yenmez and Yildirim [33] and Kominers and Sönmez [18]. Another stream of papers closer to our motivation consider affirmative action is an objective per se, formalized either by the existence of quotas as in Abdulkadiroğlu [34], Abdulkadiroğlu and Sönmez [35] and Hafalir, Yenmez and Yildirim [33], or bounds as in Ehlers [36], Ehlers, Hafalir, Yenmez, and Yildirim [37], Fragiadakis and Troyan [38] and Bó [39]. One of the objectives of these papers is to refine stability concepts and the deferred acceptance algorithms to conform to the bounds and quotas. Echenique and Yenmez [40], Erdil and Kumano [41], and Biró, Klijn and Pápai [42] consider diversity as an objective of the school district and explore the tension between diversity objectives, stability, and efficiency of priority systems and matching rules. Nguyen and Vohra [28] have a different approach and use Scarf's Lemma to implement proportional distributional constraints.

Our analysis also bears a close connection to exchange markets with balanced constraints recently studied by Dur and Uňver [2] and Biró, Klijn, and Pápai [42]. These papers model exchange programs (like the tuition exchange for children of faculty members or the Erasmus exchange program in European universities) where a balance must be kept between the number of incoming and outgoing students. Both papers consider a two-sided (rather than three-sided) matching problem, where colleges have preferences over students, and students have preferences over colleges. In the tuition exchange model of Dur and Uňver [2], students are ranked inside each college according to an exogenous priority (for example, the length of tenure of a faculty member). Dur and Uňver [2] show that balancedness may lead to impossibility results, when associated with different natural axioms, like individual rationality or fairness. Their analysis focuses on efficiency and they propose a new procedure based on the Top Trading Cycle algorithm (rather than the Deferred Acceptance algorithm). Biró, Klijn and Papai [42] also focus attention on an extension of the TTC algorithm to analyze student exchange programs where a balancedness condition holds.

**Social housing**. Finally, we note that the assignment of social housing that motivated our study has recently been analyzed in a series of papers (Leshno [43], Bloch and Cantala [44], Schummer [45] and Thakral [46]) which focus on very different aspects of the problem the revelation of persistent information on types in Leshno [43], the dynamic sequence of decisions in Bloch and Cantala [44], the manipulation of orders in Schummer [45] and multiple waiting list mechanisms in Thakral [46].

**Nested conditions**. As far as we are aware, we are the first one to deal with a pattern of perfect complements in matching markets and study the nesting of two Deferred Acceptance algorithm to tie the acceptance of agents on the three sides of the market; although we are not the only ones that nest assignments procedures. Delacrétaz, Kominers, and Teytelboym [4] generalize distributional constraints models by considering multidimensional features for the allocation of families in different localities. Such multidimensional modeling captures the services that localities can provide to families, where housing is a special case. Their problem casts similarities with ours since localities act as an intermediary between services and families. Although our model only focuses on housing, our approach is more general in the sense that apartments are shared resources for the set of localities, which is not the case in the problem of resettlement refugees. They propose the Top Choice Algorithm to generate a stable allocation in their problem. Although such mechanism in its second phase nest another algorithm, it is different to the NDA since such nested process pursues the elimination of contracts between families and localities instead of assigning households to apartments, as it is in our case. Even more, when the second phase finds a stable outcome, the housing allocation is done in a third phase which does not update the results of the second phase (as the NDA does); this last assumption reflects the fact that localities have exclusive property rights and use over apartments. Given that stable assignments may not exist, they search for quasi-stable assignment which requires that families with the lowest priority cannot block the assignment. They propose the priority focused DA under which rejection depends on being the lowest priority family. Such a mechanism is manipulable, the problem is fixed by including a maximum rank, which is an additional criterion to guarantee rejections from the top to the bottom of localities priorities. This mechanism, the MRDA, also has a phase where an algorithm is nested to update maximum ranks, which determine rejection chains. Hence, this algorithm nests processes to determine how localities should reject families instead of assigning housing among institutions, as it is the case in the NDA.

The rest of the paper is organized as follows. In the next Subsection, we discuss the relation of our model to the existing literature. Section 2 introduces the model. Section 3 presents preliminary results on the existence of fair assignments and introduces the over-demand condition. Section 4 describes the Nested DA algorithm, and Section 5 contains our main results. We show in an online appendix that it also applies to settings with maximum quotas, as in Kamada and Kojima [6]. Section 6 contains our concluding comments.

## 2. The Model

A matching market with institutions is an 8−tuple $(I, Q, H, \tau, A, P, \succ, \pi)$ where:

1. $I = \{1, 2, ..., N\}$ is a finite set of institutions, a generic institution is $i$;
2. $Q = (q_i)_{i=1}^{N}$ is a vector of quotas, where $q_i \in \mathbb{N}$ is the quota of institution $i$, and a generic quota is $q$;

3. $H = \{h_1, ..., h_H\}$ is the finite set of households, a generic household is $h$;

4. $\tau : H \to I$ is a type function, which assigns to every household an institution $\tau(h)$. Conversely, $H_i = \{h \in H \mid i = \tau(h)\}$ is the set of agents attached to institution $i$. We assume in most of the analysis that $\tau(i)$ is a function –every agent is attached to a single institution. We consider the general case where $\tau(i)$ is a correspondence –households can be attached to multiple institutions– in subsection 5.3.

5. $A = \{a_1, ..., a_A\}$ is the finite set of apartments, a generic apartment is $a$;

6. $P = (P_{h_1}, ..., P_{h_H})$ is a profile of households' preferences. For each household $h$, $P_h$ is the preference list of household $h$ over $A \cup \{\varnothing\}$, where $\varnothing$ represents that $h$ can get no apartment. We assume that $aP_ha'$ means that households $h$ strictly prefers $a$ to $a'$. We say that an apartment $a$ is acceptable for $h$ if $aP_h\varnothing$. Let $R_h$ be the antisymmetric preference list of $h$, i.e., $aR_ha'$ and $a'R_ha$ if and only if $a = a'$;

7. For all $i \in I$, $\succ^i$ is the priority of institution $i$ over sets of pairs $(a, h)$ such that $h \in H_i$. Each institution has priorities over the assignment of apartments to its households. We write $\{(a, h)\} \succ^i \varnothing$ when the assignment $(a, h)$ is acceptable for institution $i$; under this situation, we sometimes say that the pair $(a, h)$ is acceptable for $i$, which we represent by $(a, h) \succ^i \varnothing$. Moreover, $a$, or $h$, are acceptable for $i$ when it belongs to an acceptable pair of $i$.

   We assume that for all $i \in I$ the priority $\succ^i$ is responsive on elements in $2^{A \times H_i}$, i.e. for all subsets of pairs $U \in 2^{A \times H_i}$ and all pairs $(a_r, h_r)$, $(a_s, h_s) \in (A \times H_i) \setminus U$ we have that

   i. $U \cup \{(a_r, h_r)\} \succsim^i U \cup \{(a_s, h_s)\}$ if and only if $\{(a_r, h_r)\} \succsim^i \{(a_s, h_s)\}$, and

   ii. $U \cup \{(a_r, h_r)\} \succ^i U$ if and only if $\{(a_r, h_r)\} \succ^i \varnothing$.

   This general formulation allows for different interpretations, priorities may arise from scoring systems, common practice in social housing. Institutions may be benevolent, and inherit the preferences of the agents, so that $\succ^i$ is obtained directly from the preferences $P_h$ of $h \in H_i$. Alternatively, institutions may have their own fixed set of priorities, for example prioritizing among agents, or aiming at matching apartments to households with fixed characteristics independent of the preferences $P_h$, like the size of the household.

8. $\pi = (\pi_a)_{a \in A}$ is the profile of priorities of apartments over institutions; $\pi_a$ is the priority of apartment $a$ over institutions $i \in I$. Let $A_i = \{a | i\pi_a i' \text{ for all } i' \in I\}$ be the set of apartments where institution $i$ has top priority.

When matching markets are intermediated by institutions, matchings are not simply defined as two-sided matchings between households and apartments, but as three-sided matchings associating households, institutions and apartments. The mechanism we propose consists in two separate assignments: we first assign apartments to institutions, then assign households to pairs consisting of one apartment and its matched institution. We thus describe an assignment as: (i) a many-to-one matching between apartments and institutions, and (ii) a one-to-one matching between households and pairs composed by one apartment and one institution. These assignments are formalized in the following definition.

An **assignment** $\mu = (\theta, \varphi)$ is a pair such that:

i. $\theta : A \cup I \to 2^A \cup I \cup \{\varnothing\}$ where

  i.a $\theta(a) \in I \cup \{\varnothing\}$,

  i.b $\theta(i) \in 2^A$,

  i.c $a \in \theta(i)$ if and only if $\theta(a) = i$;

ii. $\varphi : (A \times I) \cup H \to (A \times I) \cup H \cup \{\varnothing\}$, where

  ii.a $\varphi(h) \in A \times I \cup \{\varnothing\}$,

  ii.b $\varphi(a, i) \in H \cup \{\varnothing\}$,

  ii.c $\varphi(h) = (a, i) \Leftrightarrow \varphi(a, i) = h$.

iii. $\theta(a) = i$ if and only if $\varphi(h) = (a, i)$ for some $h \in H$.

If $\mu = (\theta, \varphi)$ is an assignment, we also use the notation $\varphi(h) = (\varphi_A(h), \varphi_I(h))$ when $\varphi(h) = (a, i)$. That is to say, $a = \varphi_A(h)$ and $i = \varphi_I(h)$.

Conditions i. a, b and c define the many-to-one matching $\theta$ between apartments and institutions. Conditions ii. a, b and c define the one-to-one matching $\varphi$ between households and pairs composed by one apartment and one institution. Condition iii. defines a consistency condition between the two matchings, by requiring that whenever a household is assigned to a pair consisting of an apartment and an institution in $\varphi$, the apartment and institution are assigned to each other in $\theta$.

The match of $h \in H$ is $\varphi(h) \in (A \times I) \cup \{\varnothing\}$, $h$ is unmatched if $\varphi(h) = \varnothing$. The assignment of $i$ is $\mu(i) = \{(a, h) \in A \times H : a \in \theta(i) \text{ and } \varphi_A(h) = a\}$.

We illustrate the definition of the assignments $\theta$ and $\varphi$ with the following example. Consider $(I, Q, H, \tau, A, P, \succ, \pi)$ a market with institutions where $I = \{1, 2, 3\}$, $H = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$, a vector of priorities $\succ$, a vector of preferences $P$ and a profile of priorities $\pi$. The vector of quotas is $Q = (1, 2, 1)$, and the type correspondence is given by $H_1 = \{h_3\}, H_2 = \{h_4, h_5\}$, and $H_3 = \{h_1, h_2\}$. A typical assignment for this market is represented as follows

$$\mu = \begin{pmatrix} h_3 & h_5 & h_1 & \varnothing & h_2 & h_4 & \varnothing \\ \overbrace{\qquad}^{\varphi(h_3)} & \overbrace{\qquad}^{\varphi(h_5)} & \overbrace{\qquad}^{\varphi(h_1)} & \underbrace{\varnothing}_{\varphi^{-1}(\varnothing)} & \overbrace{\qquad}^{\varphi(h_2)} & \overbrace{\qquad}^{\varphi(h_4)} & \overbrace{\qquad}^{\varphi(\varnothing)} \\ a_2 & a_4 & a_3 & & \varnothing & \varnothing & a_1 \\ 2 & 2 & 3 & 1 & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$
$$\underbrace{\qquad\qquad}_{\theta(2)} \quad \underbrace{\quad}_{\theta(3)} \quad \underbrace{\quad}_{\theta(1)} \quad \underbrace{\qquad\qquad\qquad}_{\theta^{-1}(\varnothing)}$$

In this assignment, institution 1 ends up with no apartment, and hence does not fulfill its quota. Institution 2 is assigned the two apartments $a_2$ and $a_4$ which are given to households $h_3$ and $h_5$, and the quota is fulfilled. Institution 3 also fills its quota, obtaining apartment $a_3$ which is assigned to household $h_1$. Apartment $a_1$ remains unassigned as well as households $h_2$ and $h_4$.

We now define the choice functions of institutions. Consider sets of the form $\mathcal{U}_i = \{U \in 2^{A \times H_i} |$ neither households nor apartments are paired twice in $U\}$. The set $\mathcal{U}_i$ collects pairs of apartments and households attached to institution $i$ such that every apartment and household only appear in one of the pairs. For any institution $i \in I$, we define the **choice function** $Ch_i$ as a mapping choosing the pairs with the highest priority for $i$ in $\mathcal{U}_i$: for all $(U, q_i) \in 2^{A \times H_i} \times \mathbb{Z}_+$, the choice of $i$ is the set $Ch_i(U, q_i) = \max_{\succ^i}\{u \subseteq U \mid |u| \leq q_i \text{ and } u \in \mathcal{U}_i\}$.

We now extend classical properties of the assignment $\mu$ to matching with institutions. An assignment $\mu$ is **individually rational** if

i. for all $h \in H$ either $\varphi_A(h) P_h \varnothing$ and $\varphi_I(h) = \tau(h)$,[5] or $\varphi(h) = \varnothing$, and

ii. $\mu(i) = Ch_i(\mu(i), q_i)$.[6]

In words, an assignment is individually rational if both households and institutions –the two sides of the market which are endowed with preferences and priorities– prefer the outcome of matching $\mu$ to what they would get by not participating in the matching, $\varnothing$. An individually rational assignment $\mu = (\theta, \varphi)$ is **feasible if** $|\theta(i)| = q_i$ for all $i \in I$. Hence, an assignment satisfies the **distributional constraints** when the assignment is feasible.

An assignment $\mu$ is **non-wasteful** if no household-institution pair $(h, i)$ can claim an empty apartment $a$, i.e. there is no $i$, $h$ and $a$ such that:

i. $aP_h\varphi_A(h)$,

ii. $(a, h) \in Ch_i(\mu(i) \cup \{(a, h)\}, q_i)$, and

iii. $\theta(a) = \varnothing$.

A household-institution pair $(h, i)$ has **justified envy** over the household-institution pair $(h', i')$ at the individually rational assignment $\mu$ if $i = \tau(h)$, $i' = \tau(h')$ and there exists $\varphi_A(h') = a \in \theta(i')$, such that

i. $aP_h\varphi_A(h)$,

ii. $(a, h) \in Ch_i(\mu(i) \cup \{(a, h)\}, q_i)$, and

iii. $i\pi_a i'$ or $i = i'$.

We thus define deviations from pairs $(h, i)$, where $(h, i)$ claims an apartment $a$ that improves preferences of both household $h$ and institution $i$ and for which institution $i$ has higher priority than institution

[5]For the general case, when $\tau$ is a correspondence, we need that $\varphi_I(h) \in \tau(h)$

[6]Since priorities are responsive, this means that, for all $i \in I$, either $(a, h) \succ^i \varnothing$ for all $(a, h) \in \mu(i)$, or $\mu(i) = \varnothing$.

$i'$ which is currently assigned to apartment $a$. Notice that we require that the new assignment obtained after the deviation satisfies the quota for the deviating institution. Nevertheless, the assignment following the deviation may not satisfy the distributional constraints, when the apartment $a$ is initially assigned to an institution $i'$ different from $i$. If we consider only deviations which do not require a change in assignment of apartments to institutions – and hence allows us to compare assignments which both satisfy the distributional constraints – we obtain a weaker form of envy, called justified envy over households of the same type:

There is **justified envy over households of the same type** when a pair $(h, i)$ has justified envy over a pair $(h', i)$ at an IR assignment $\mu$. This concept searches for justified envy within those pairs $(a, h)$ intermediated by institution $i$. Hence, when $\tau$ is a correspondence, it is not necessary that $\tau(h)$ be equal to $\tau(h')$, justified envy over households of the same type may arise by having $\tau(h) \cap \tau(h') \neq \varnothing$.

We now define fair and efficient assignments. An assignment $\mu$ is **fair** if it is individually rational, non-wasteful and there is no justified envy. A matching $\mu$ is **fair over households of the same type** if it is individually rational, non-wasteful and there is no justified envy for households of the same type.

An assignment $\mu$ is **Pareto efficient** if there is no matching $\mu'$ such that all households weakly prefer $\mu'$ to $\mu$, with strict preference for at least one household. An assignment $\mu'$ **Pareto dominates** another assignment $\mu$ if $\mu'(h) R_h \mu(h)$ for each $h \in H$, and $\mu'(h') P_{h'} \mu(h')$ for at least one $h' \in H$.

Finally we consider the incentives of households to reveal their true preferences: A **mechanism** $\Lambda$ associates a profile of preference lists with an assignment $\mu$. Let $R_h$ be the true preference list of each household $h$. The set of all possible preference lists of household $h$ is denoted by $\Re_h$. A profile of preference list is a vector $R' = (R'_{h_1}, R'_{h_2}, \ldots, R'_{h_H}) \in \Re_{h_1} \times \Re_{h_2} \times \cdots \times \Re_{h_H} = \Re$. As usual, $R_{-h}$ is the profile of all preference lists except $R_h$.

A mechanism $\Lambda$ is **strategy-proof for households** if telling the truth is a dominant strategy for all households, i.e. $\Lambda[R_h, R_{-h}](h) R_h \Lambda[R'_h, R_{-h}](h)$ for all $R'_h \in \Re_h$ and $R_{-h} \in \Re_{-h}$.

## 3. The Nested Deferred Acceptance Mechanism

In the matching market through institutions, we care about fairness since households care about apartments, but the household-apartment allocation requires the intermediation of an institution given that $A$ is a set of shared resources. Hence, as in Gale and Shapley [47], we are interested in generating a fair assignment between households, institutions, and apartments. So, we extend the Deferred Acceptance (DA) algorithm to our setting by introducing the Nested Deferred Acceptance (NDA), which produces an assignment $\mu = (\theta, \varphi)$ in the matching market through institutions.

The idea behind the NDA is to compute simultaneously a many-to-one matching, $\theta$, and a one-to-one matching, $\varphi$, by nesting two deferred acceptance algorithms. In the main DA iteration (the "outer loop "), each unassigned household asks for her most preferred apartment. Given these demands, we run another DA (the "inner loop") where each institution demands a set of apartments and assigns them.

The procedure continues iteratively. Here, we present a general overview of the NDA to get insights into the previous explanation; the full details of this algorithm are in Appendix Appendix A.

**Initialization**

Consider a market $(I, Q, H, \tau, A, P, \succ, \pi)$. The assignment is initialized to be the empty assignment, i.e., each household, institution, and apartment is unassigned at the beginning of the assignment procedure.

**$\mathbf{A}^t$. Outer loop**

At step $t$, households ask for the preferred apartment among those which have not been rejected them, while matched households iterate their demand to their match. This phase corresponds to the outer loop, and its objective is to elicit the demand of households.

Each institution observes the apartments that its households demand and elaborate all possible household-apartment pairs that they can intermediate at step $t$. Each institution only focuses on getting the use of acceptable apartments. Then, the inner loop starts.

**$\mathbf{B}^t$. The inner loop**

In this phase, institutions compete among them to get a set of apartments belonging to household-apartment pairs that are maximal according to their priority orders. So, this loop is a DA where institutions ask for a set of apartments with the highest priority under $\succ$ (step **$\mathbf{B}^t$.1**), and apartments reject institutions following their priorities (step **$\mathbf{B}^t$.2**).

When each institution has fulfilled its quota, or is not acceptable by any other apartment, the inner loop has found a tentative assignment between institutions and apartment. In other words, institutions are tentatively assigned to a set of apartments, not necessarily the ones that they own. Then, each apartment is tentatively assigned to a household if and only if the set of pairs $(a, h)$ is maximal under the priority $\succ^i$ (step **$\mathbf{B}^t$.3**).

**$\mathbf{C}^t$. Ending conditions**

The tentative assignment, produced at the end of step **$\mathbf{B}^t$.3**, indicates the end of the inner loop. The algorithm goes to the outer loop, phase $\mathbf{C}^t$. In this phase, each household updates the set of apartments that have not rejected them. If every household is assigned to some apartment, or have been rejected by all her acceptable apartments, the NDA stops. Otherwise, go to phase $\mathbf{A}^{t+1}$.

The output of the previous mechanism depends on the market $E = (I, Q, H, \tau, A, P, \succ, \pi)$. So, it is denoted by $\mu^{NDA}[E] = (\theta^{NDA}[E], \varphi^{NDA}[E])$, or simply $\mu^{NDA} = (\theta^{NDA}, \varphi^{NDA})$ whenever there is no confusion. We use $NDA[P]$ to denote the NDA algorithm under the preference profile $P$. Note that the NDA algorithm has a finite number of steps because each DA ends in finite time.

### *3.1. An example of NDA*

The following example shows how the NDA algorithm works.

**Example 3.1.** Consider the market $I = \{1, 2\}$, $H = \{h_1, h_2, h_3\}$ and $A = \{a_1, a_2\}$. The vector of quotas is $Q = (1, 1)$, and the type function is defined by $H_1 = \{h_1\}$ and $H_2 = \{h_2, h_3\}$. The profiles of institutions' priorities, households' preferences, and apartments' priorities are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1, h_1) & (a_1, h_2) \\ (a_2, h_1) & (a_2, h_3) \\ & (a_2, h_2) \end{pmatrix}, \; P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} \\ a_1 & a_1 & a_2 \\ a_2 & a_2 & a_1 \end{pmatrix} \text{ and } \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} \\ 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

At step $\mathrm{A}^1.1$, households announce their preferred apartment: households $h_1$ and $h_2$ announce $a_1$, and household $h_3$ announces $a_2$. The NDA then moves to step $\mathrm{B}^1.1$, which is the inner loop. At this step, each institution chooses the set of apartment-household pairs with the highest rank at its priority's list: 1 announces $\{(a_1, h_1)\}$ and 2 announces $\{(a_1, h_2), (a_2, h_3)\}$. Such announcement indicates that apartment $a_1$ is demanded by both institutions, so the algorithm moves to step $\mathrm{B}^1.2$. As institution 1 has priority over 2 for apartment $a_1$, $a_1$ rejects 2 and the assignment at the end of step $\mathrm{C}^1.1$ is $(a_1, h_1), (a_2, h_3)$.

The NDA moves back to the outer loop and household $h_2$ announces apartment $a_2$ at phase $\mathrm{A}^2.2$. The algorithm moves now to $\mathrm{B}^2.1$ where institution 2 announces that it prefers $(a_2, h_3)$ to $(a_2, h_2)$, and the pairs $(a_1, h_1), (a_2, h_3)$ are matched. This is the last step of the algorithm as household $h_2$ has already announced all the apartments.

Example 3.1 also allows us to show the importance of the inner loop (Phase $\mathrm{B}^t$). Notice, in the absence of the inner loop, household $h_3$ is rejected from 2 at the end of step $\mathrm{A}^1$. Thus, the regular DA stops at the end of $\mathrm{A}^2$, and the algorithm outputs the assignment $(a_1, h_1), (a_2, h_2)$. In this assignment, note that $(h_3, 2)$ has justified envy over $(h_2, 2)$; that is to say, the DA procedure is not fair in the matching through institutions model.

### *3.2. No distributional constraints*

In this section we consider a hypothetical benchmark where there are no distributional constraints, or they are ineffective; formally, there are no distributional constraints when institutions' quotas are large enough, i.e., $q_i > |A|$ for all $i \in I$. The implications for the NDA mechanism are presented below; we first show that the "inner loop" (phase $\mathrm{B}^t$) of the NDA algorithm only requires one step.

**Lemma 3.1.** *Consider a matching market with institutions and no distributional constraints. Then, phase $B^t$ of the NDA algorithm is iterated only once.*

In the absence of distributional constraints, if an apartment is tentatively assigned to a household and institution at some step during the algorithm, it is assigned to some household-institution pair in the final assignment, not necessarily the same household-institution pair.

**Lemma 3.2.** *Consider a matching market with institutions and no distributional constraints. If an apartment is assigned at some step $t$ by some institution, this apartment is assigned under the assignment $\mu^{NDA}$.*

The following theorem shows all the desirable properties that the NDA algorithm satisfies when there are no distributional constraints, i.e., when we consider that .

**Theorem 3.1.** *Consider a matching market with institutions* $(I, Q, H, \tau, A, P, \succ, \pi)$ *with no distributional constraints.*

1. *The* $\mu^{NDA}$ *assignment is individually rational, non-wasteful and there is no justified envy; namely, the assignment* $\mu^{NDA}$ *is fair.*
2. *There is no fair assignment that Pareto dominates* $\mu^{NDA}$.
3. *The NDA mechanism is strategy-proof for households.*

Theorem 3.1 shows that in the absence of distributional constraints, the NDA in the model of matching through institutions inherits the properties of DA in classical school choice problems. Even though the description of assignments is more complex in matching through institutions than in classical school choice problems, the properties of the assignment rule do not differ significantly.

Notably, by pooling apartments, we observe that the institutions ensure to their households an assignment at least as good as in autarky by assigning to themselves top priorities on their objects. In this sense, the NDA mechanism complies with the participation constraints criteria.

## 4. Existence issues with effective distributional constraints

As it is common in the literature of distributional constraints, feasible assignments may not exist. Moreover, if feasible assignments exist, it is possible that satisfying distributional constraints may lead to inconsistencies in terms of fairness. In this section we discuss existence issues.

### *4.1. Distributional constraints*

The following example presents a matching market through institutions where there is not a feasible assignment.

**Example 4.1.** Let $I = \{1, 2\}$, $A = \{a_1, a_2, a_3\}$, $H = \{h_1, h_2, h_3, h_4\}$, where households type function is given by $H_1 = \{h_1, h_2, h_3\}$ and $H_2 = \{h_4\}$. The vector of quotas is $Q = (2, 1)$. The profiles of institutions' priorities and households' preferences are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1, h_1) & (a_3, h_4) \\ (a_3, h_2) & \end{pmatrix}, \qquad P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} \\ a_1 & a_1 & a_1 & a_1 \\ a_2 & a_3 & a_3 & a_2 \\ a_3 & a_2 & a_2 & a_3 \end{pmatrix}.$$

Notice that there is no assignment of apartment $a_2$ which is acceptable by any institution. This means that apartment $a_2$ cannot be assigned in an individually rational assignment and quotas cannot be fulfilled. Therefore, there are no feasible assignment for this market. □

Even more, feasible assignments may not exist even if all apartments are acceptable for at least one institution and the number of attached households is greater than the quota of each institutions. This situation is shown in the following example.

**Example 4.2.** Let $I = \{1, 2\}$, $\{a_1, a_2, a_3\}$, $H = \{h_1, h_2, h_3, h_4, h_5\}$, where households type function is given by $H_1 = \{h_1, h_2, h_3\}$ and $H_2 = \{h_4, h_5\}$. The vector of quotas is $Q = (2, 1)$. The profiles of institutions' priorities and households' preferences are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1, h_1) & (a_1, h_4) \\ (a_2, h_1) & (a_3, h_5) \\ (a_3, h_1) & \end{pmatrix}, P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} & P_{h_5} \\ a_1 & a_1 & a_1 & a_1 & a_1 \\ a_2 & a_3 & a_3 & a_2 & a_2 \\ a_3 & a_2 & a_2 & a_3 & a_3 \end{pmatrix}.$$

In this case, there is no assignment that respects distributional constraints since institution 1 only cares about household $h_1$. □

In Example 4.1, the fulfillment of quotas would imply to assign an unacceptable apartment. This situation prevails even if all apartments are acceptable for at least one institution and the number of households attached to each institution is greater than the institution's quota, as we observe in Example 4.2.

*4.2. Fairness*

Even when individually rational assignments satisfying distributional constraints exist, they may not satisfy fairness due to the priorities of institutions. We illustrate this point in the following example.

**Example 4.3.** Consider $I = \{1, 2\}$, $H = \{h_1, h_2, h_3\}$ and $A = \{a_1, a_2, a_3\}$. The vector of quotas is $Q = (q^1, q^2) = (2, 1)$, and the type function is given by $H_1 = \{h_1, h_2\}$ and $H_2 = \{h_3\}$. The profiles of institutions' priorities, households' preference and apartments' priorities are

$$\begin{pmatrix} \succ^1 & \succ^2 \\ (a_2, h_2) & (a_1, h_3) \\ (a_1, h_1) & \\ (a_1, h_2) & \\ (a_3, h_1) & \end{pmatrix}, \quad P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} \\ a_1 & a_1 & a_1 \\ a_2 & a_2 & a_3 \\ a_3 & a_3 & a_2 \end{pmatrix}, \quad \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} & \pi_{a_3} \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

Notice that $(a_1, h_3)$ is the unique acceptable assignment of institution 2 and $(a_2, h_1)$ is not acceptable for institution 1. Therefore, in this market there exists one and only one assignment that satisfies the distributional constraints, which is:

$$\mu = \begin{pmatrix} h_1 & h_2 & h_3 \\ a_3 & a_2 & a_1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Although $\mu$ satisfies the distributional constraints, note that i) $a_1 P_{h_1} a_3$, ii) $(a_1, h_1) \in Ch_1(\mu(1) \cup \{(a_1, h_1)\}, q_1)$ because $(a_1, h_1) \succ^1 (a_3, h_1)$, and iii) $1\pi_{a_1}2$. Hence, the pair $(h_1, 1)$ has justified envy over the pair $(h_3, 2)$ at the apartment $a_1$. Consequently, the assignment $\mu$ is not fair. □

The previous examples build on the fact that distributional constraints may conflict with the preferences of households and the priorities of institutions. In Example 4.3, the unique assignment which satisfies distributional constraints is upset by a deviation which results in one of the two institutions not fulfilling its quota.

Assuming that every apartment is acceptable to any household and institution would clearly overcome the difficulties highlighted by Examples 4.1, 4.2 and 4.3. However, it is possible to provide a weaker requirement which guarantees the existence of a fair assignment satisfying distributional constraints. Specifically, it is enough to assume that for each institution and each apartment there exist a household who is willing to accept the apartment, and such that the institution also accepts the assignment. This leads us to define an over-demand condition, under which fair assignments satisfying the distributional constraints will be shown to exist.

**Assumption 1.** (**Over-demand Condition**) For all IR assignments, for all institutions $i$ and apartments $a$, there is an unassigned household $h$ such that $(a, h) \succ^i \varnothing$, $aP_h\varnothing$ and $i\pi_a\varnothing$.

This condition implies the existence of at least one IR assignments that satisfies the distributional constraints. Even more, the previous condition does not rely on the market size, but if a market satisfies this condition, it is large.

*4.3. No lattice structure*

Finally, we note that households may disagree on the best fair assignment satisfying distributional constraints.

**Example 4.4.** Consider a market such that $H = \{h_1, h_2\}$, $A = \{a_1, a_2\}$ and $I = \{1, 2\}$, $H_1 = \{h_1\}, H_1 = \{h_2\}$. Institutions priorities, households' preferences and apartments' priorities are given by.

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1, h_1) & (a_2, h_2) \\ (a_2, h_1) & (a_1, h_2) \end{pmatrix}, P = \begin{pmatrix} P_{h_1} & P_{h_2} \\ a_1 & a_1 \\ a_2 & a_2 \end{pmatrix} \text{ and } \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} \\ 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

In this market, we can find the following fair assignments that cope with the distributional constraints.

$$\mu = \begin{pmatrix} h_1 & h_2 \\ a_1 & a_2 \\ 1 & 2 \end{pmatrix} \text{ and } \mu' = \begin{pmatrix} h_2 & h_1 \\ a_1 & a_2 \\ 2 & 1 \end{pmatrix}.$$

Let $\Gamma$ be the set of all fair assignments that fulfill the distributional constraints. Considering $\mu_1, \mu 2 \in \Gamma$, we define the order $\succ_H$ over $\Gamma$ as follows,

$$m_1 \succsim_H \mu_2 \text{ if and only if } \mu_1(h) R_h \mu_2 \mu_2(h) \text{ for all } h \in H.$$

In Example 4.4 we have that household $h_1$ prefers $\mu'$ to $\mu$, whereas household $h_2$ has the reverse preferences. Therefore, $\Gamma$ does not have a lattice structure under $\succ_H$ .

## 5. Dealing with distributional constraints

The presence of distributional constraints may prevent the NDA from producing fair outcomes. During Phase $B^t$ of the NDA algorithm, institutions make offers following the preferences of households attached to them, and they might temporarily fill their quotas. Later in the run of the NDA, better options might arise for some institutions, leading them to drop some apartments. In this case, institutions act as interrupters, which temporarily accept household-apartments pairs that will be dropped in the final outcome, meanwhile the access to other institutions is prevented and thus the emergence of fair outcomes.

### *5.1. Interrupters*

The following example shows that the NDA may produce an assignment which fails to satisfy fairness. We show that such problem arises given the existence of institutions that made *interruptions*, which generates justified envy over households of the same type.

**Example 5.1. (There is justified envy over households of the same type).** Let $I = \{1, 2\}$, $A = \{a_1, a_2, a_3\}$ and $H = \{h_1, h_2, \ldots, h_7\}$, where households type function is given by $H_1 = \{h_1, h_2, h_5, h_6\}$ and $H_2 = \{h_3, h_4, h_7\}$. The vector of quotas is $Q = (2, 1)$. The profiles of institutions' priorities and households' preferences are

$$\succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (a_1,h_1) & (a_2,h_3) \\ (a_1,h_2) & (a_2,h_4) \\ (a_2,h_1) & (a_1,h_3) \\ (a_2,h_2) & (a_1,h_4) \\ (a_1,h_5) & (a_1,h_7) \\ (a_2,h_6) & (a_2,h_7) \\ (a_3,h_1) & \\ (a_3,h_6) & \end{pmatrix}, \qquad P = \begin{pmatrix} P_{h_1} & P_{h_2} & P_{h_3} & P_{h_4} & P_{h_5} & P_{h_6} & P_{h_7} \\ a_1 & a_2 & a_1 & a_1 & a_2 & a_1 & a_1 \\ a_2 & a_1 & a_2 & a_2 & a_1 & a_2 & a_2 \\ a_3 & a_3 & a_3 & a_3 & a_3 & a_3 & a_3 \end{pmatrix}.$$

The profile of apartments' priorities is $\pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} & \pi_{a_3} \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

Running the NDA algorithm (see the Appendix Appendix B), we get the following assignment

$$\mu^{NDA} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_3 & a_1 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & 1 & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

Note that the sets $\{(a_1,h_5),(a_2,h_6),(a_3,h_3)\}$ and $\{(a_1,h_4),(a_2,h_4),(a_1,h_7),\ (a_2,h_7)\}$ are sets of unassigned pairs that are acceptable for institutions 1 and 2, respectively. Consequently, the market is over-demanded. Moreover, note that $a_1P_{h_1}a_3$ and $\{(a_1,h_1)\} \succ^1 \{(a_1,h_2)\}$, where $\varphi_I^{NDA}(h_1) = \varphi_I^{NDA}(h_2) = 1$, $\varphi_A^{NDA}(h_2) = a_1$ and $\varphi_A^{NDA}(h_1) = a_3$. Thus, the pair $(h_1, 1)$ has justified envy over $(h_2, 1)$ at apartment $a_1$. In other words, $\mu^{NDA}$ is not fair. $\square$

Example 5.1 shows a market where the NDA fails to generate a fair assignment, even when the over-demand condition holds. Note that the assignment is not fair because there is justified envy over households of the same type; we observe that such a problem arises due to an interruption made by one institution. This example allows us to describe how an interrupter works. Running the NDA (all steps are developed in Appendix Appendix B), institution 2 is tentatively assigned to $a_1$ at step 1. By this tentative allocation, household $h_1$ and institution 1 are displaced from $a_1$ at the end of step 1. Similarly, household $h_3$ and institution 2 are displaced from apartment $a_1$ at step 2 (see Table B.4 in the Appendix Appendix B). Later, at step 3, institution 1 asks for $a_1$ and gets it because $1\pi_{a_1}2$. However, note that $h_1$ can no longer demand this apartment because she was rejected from it at step 1. As a consequence, the choice function of institution 1 does not consider the pair $(a_1, h_1)$, which implies the presence of justified envy between the pairs $(h_1, 1)$ and $(h_2, 1)$.

Summarizing the previous observation, we have that apartment $a_1$ is tentatively assigned to institution 2 during steps 1 and 2 of the NDA, but $a_1$ does not belong to $\theta(2)$ in the final assignment. In the terminology of Kesten [5], we have that institution 2 is an interrupter for apartment $a_1$. Below, we present the formal definition for an interrupter institution.

**Definition 1.** Given a matching problem to which the NDA is applied, we say that an institution $i$ is an **interrupter** for apartment $a$ if there exists

1. steps $t$ to $t+n$ such that $a \in \theta^{t'}(i)$ for all $t' \in \{t, t+1, \dots, t+n\}$ but $a \notin \theta^{t'}(i)$ for all $t' > t+n$, and
2. an institution $j \neq i$ and a household $h$ such that $(a,h) \in Ch_j(M_j^{t'}, q_j)$ but $(a,h) \notin \mu^{t'}(j)$ for some $t' \in \{t, t+1, \dots, t+n\}$.

Although the definition of interruption is the same in our case to the one in Kesten [5], the role that plays the agent who makes the interruption is different. In the cited paper, the agents that demand, students, are the ones that make the interruption and cause a loss of efficiency in the DA. In our case, institutions are the ones who make interruptions which hurts fairness. The interruption is due to the fact that institutions are intermediaries, they act as receivers of the demand of households, on which they elaborate their own demand on apartments: the interruption occurs because institutions do not make offers to apartments in the decreasing order of their priorities. In the following section, we show that the deletion of interrupters in our setting contributes to restoring fairness.

### *5.2. Nested Deferred Acceptance with Interrupters*

To overcome the problem caused by interruptions, we modify the NDA introducing a second stage where we search for all interrupter institutions, as in [5]. Then, we let these institutions delete from their priorities the pairs containing the apartment where they cause the interruption. Note that Kesten [5] defines this operation over students' preferences (the equivalent in our model of households in the outer loop), because he identifies students as the source of efficiency loss in the Deferred Acceptance algorithm. In our case, the delete operation targets institutions, which are involved in the inner loop of the algorithm, as they are the source of justified envy in the NDA algorithm.

Let $\Im$ be the set of all possible priorities $\succ^i$, for all $i \in I$. The **delete operation** over $\Im$ is the function $\backslash : \Im \times (A \times H) \to \Im$ such that $\backslash(\succ, a)$, or simply $\succ \backslash a$, is the priority that declares all pairs $(a, h) \succ^i \varnothing$ as unacceptable for $i$. In other words, the priority $\succ \backslash a$ drops all acceptable pairs $(a, h)$ from $\succ^i$ and holds the original order in the priority $\succ^i$.

Now, we formally present the Nested Deferred Acceptance with Interrupters (NDAI). Each step of this mechanism has two stages: the NDA algorithm runs in the first stage, while the second stage deletes pairs from the priorities of interrupters. The NDAI proceeds as follows.

**Initialization** Initialize the counter of iterations over interrupter institutions at $x := 0$.

**Step 0**. This step is divided in the following stages:

**Stage 0.1 NDA Phase**. Let $\succ^0 = (\succ^{i,0})_{i \in I} = (\succ^i)_{i \in I}$. Run the NDA algorithm using the profile of priorities and preferences $(\succ^0, P)$.

**Stage 0.2 Deletion in Priorities** If there are no interrupters at Stage 0.1, the algorithm stops. Otherwise, find the last step of the NDA phase at which an interrupter is rejected from the apartment for which it is an interrupter. For each interrupter institution $i$, $\succ^{i,1} = \succ^{i,0} \setminus a$; $\succ^{j,1} = \succ^{j,0}$ if $j$ is not an interrupter.

**Step** $x$. This step is divided in the following stages:

**Stage x.1 NDA Phase.** Run the NDA algorithm with the profile of priorities and preferences $(\succ^x, P)$.

**Stage x.2 Deletion in Priorities.** If there are no interrupters at Stage x.1, the algorithm stops. Otherwise, find the last step of the NDA phase at which an interrupter is rejected from the apartment for which it is an interrupter. For each interrupter institution $i$, $\succ^{ix+1} = \succ^{i,x} \setminus a$; $\succ^{j,(x+1)} = \succ^{j,x}$ if $j$ is not an interrupter.

The output of the previous mechanism is denoted by $\mu^{NDAI}$. The NDAI is solvable in finite time because each NDA phase is solvable in a finite number of steps, and there are at most $|I|$ interrupters.

To illustrate how the NDAI algorithm works, we continue with the analysis of Example 5.1, which shows the existence of interrupters.

**Example 5.2.** We consider the same market as in the Example 5.1, in which we identify that institution 2 causes an interruption over the pair $(a_1, h_1)$. In other words, this example corresponds to the stage 0.1 of the NDAI algorithm. So, we delete all acceptable pairs that include $a_1$ from the priority $\succ^2$. Appendix Appendix B illustrates the complete running of the NDAI algorithm.

At stage 1.1 we observe that there is no interrupters, and the algorithm finishes. The NDAI mechanism outputs the following assignment

$$\mu^{NDAI} = \mu^4 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & a_3 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & 1 & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

Note that $\mu^{NDAI}$ is fair over households of the same type. □

Respecting to desirable properties, the NDAI is strategy proof for households, while students can manipulate the EADAM. The next Theorem shows that the assignment produced by the NDAI algorithm satisfies desirable properties.

**Theorem 5.1.** *Consider a matching market with institutions $E = (I, Q, H, \tau, A, P, \succ, \pi)$ that satisfies the over-demand condition.*

1. *The $\mu^{NDAI}$ is individually rational, non-wasteful, respects distributional constraints and there is no justified envy; namely, the assignment $\mu^{NDAI}$ is fair.*
2. *There is no assignment which is fair that Pareto dominates $\mu^{NDAI}$.*
3. *The NDAI is strategy-proof for households.*

Theorem 5.1 extends the classical properties of the deferred acceptance algorithm to the model of matching through institutions. The proof, given in the Appendix, is an adaptation to our model of classical proofs of fairness and strategy-proofness of the Deferred Acceptance algorithm. Theorem 5.1 provides a strong rationale for the use of this mechanism to assign agents to apartments when institutions face distributional constraints.

### *5.3. Multiple institutions*

In this subsection we generalize our previous results by allowing households to be attached to multiple institutions.[7] We now assume that the type assignment mapping $\tau$ is a correspondence rather than a function: $\tau : H \to 2^I$, i.e., $\tau(h) \subseteq I$ and $H_i = \{h \in H \mid i \in \tau(h)\}$. We first note that fairness is too demanding and must be weakened when agents can belong to multiple institutions. The following example, inspired by Biró and McDermid [8], illustrates why fairness may fail.

[7] This is the case for applications to social housing in Paris.

**Example 5.3.** (**No fair assignments with multiple types**). Consider $H = \{h_1, h_2\}$, $I = \{1, 2\}$ and $A = \{a_1, a_2\}$. The type function is defined as $H(1) = \{h_1, h_2\} = H(2)$. Households' preferences, institutions' priorities and apartments' priorities are

$$P = \begin{pmatrix} P_{h_1} & P_{h_2} \\ a_1 & a_1 \\ a_2 & \end{pmatrix}, \ \succ = \begin{pmatrix} \succ^1 & \succ^2 \\ (h_1, a_1) & (h_1, a_2) \\ (h_2, a_1) & \end{pmatrix} \text{ and } \pi = \begin{pmatrix} \pi_{a_1} & \pi_{a_2} \\ 1 & 2 \\ 2 & \end{pmatrix}.$$

In the previous matching market through institutions, there is only one feasible assignment; the assignment that satisfies the distributional constraints is

$$\mu = \begin{pmatrix} h_1 & h_2 \\ a_2 & a_1 \\ 2 & 1 \end{pmatrix}.$$

Now, note that $a_1 P_{h_1} a_2$, $\{(h_1, a_1)\} \succ^1 \{(h_2, a_1)\}$ and $1 \pi_{a_1} 2$. Consequently, in the assignment $\mu$ we have that pair $(h_1, 1)$ has justified envy over the pair $(h_2, 2)$ at apartment $a_1$.

Notice that the over-demand condition is satisfied in Example 5.3. Hence the inexistence of assignment satisfying fairness and distributional constraints comes from another source, here the fact that household $h_1$ belongs to the list of the two institutions which have different priorities over the apartment matched to household $h_1$. In order to preclude this phenomenon, we do not allow for envy involving households attached to two different institutions and consider the weaker requirement of fairness over households of the same type. We can then adapt Theorem 5.1 to show that the NDAI still satisfies desirable properties.

**Theorem 5.2.** *Consider a matching market with institutions $E = (I, H, \tau, Q, A, P, \succ, \pi)$ where each institution is over-demanded and households can belong to many institutions.*

1. *The $\mu^{NDAI}$ is individually rational, non-wasteful, respects distributional constraints and there is no justified envy over households of the same type; namely, the assignment $\mu^{NDAI}$ is fair over households of the same type.*
2. *There is no fair over households of the same type assignment that Pareto dominates $\mu^{NDAI}$.*
3. *The NDAI is strategy-proof for households.*

## 6. Concluding Remarks

We model a matching market with institutions as a three-sided market. Institutions have agents attached to them, and have priorities over the assignment of objects to their agents. Agents have preferences over objects. Objects have priorities over institutions. We show that fair assignments satisfying distributional constraints may fail to exist, and propose a sufficient condition – the over-demand condition – under which we prove existence. Existence derives from the construction of a new algorithm, the Nested Deferred Acceptance (NDA) algorithm, which combines a one-to-one matching between agents and objects and a one-to-many matching between objects and institutions. If interrupters are eliminated from the priority list, as in Kesten [5], the NDA algorithm produces an assignment which is fair, Pareto optimal among fair assignments and strategy-proof for agents in a problem where the set of stable assignments has no lattice structure.

The matching with contracts approach (Hatfield and Milgrom [3]) also analyses stability in settings where the set of stable matchings has no lattice structure. This approach has been successful relaxing the substitute condition by Kelso and Crawford [13] and establishing that the Generalized Gale-Shapley algorithm outputs a stable allocation, even in settings where the set of stable matchings has no lattice structure, like in Hatfield and Kojima [14], Kominers and Sönmez [18] and Hatfield and Kominers [19]. Contracts and the cumulative offer mechanism, however, are two-sided, which limits the scope of the analysis to pairwise stability, which is not the natural stability restriction in our application since our setting is closely related to a three-sided structure given that the use of objects (apartments) is not exclusive.

The model of matching through institutions we consider is inspired by the assignment of social housing in Paris but the procedure we propose applies more generally to situations where agents belong to different groups, and pool their resources to obtain more flexible outcomes. We hope to study more applications –like the exchange of students across universities or of pupils across school districts– in detail in future work.

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## Appendix

## Appendix A. The NDA algorithm

### Initialization

Consider a market $(I, Q, H, \tau, A, P, \succ, \pi)$. The assignment is initialized to be the empty assignment, so $\mu^0(i) = \mu^0(a) = \mu^0(h) = \varnothing$, i.e. $\theta^0(i) = \theta^0(a) = \varphi^0(h) = \varnothing$ for all $i \in I$, $a \in A$, $h \in H$.

Let $A_h^t = A$ and $t := 1$.

**$A^t$. Eliciting the demand of households (the outer loop)**

All unassigned households $h$ ask for their most preferred apartment in $A_h^t$, denoted by $D_h^t$, while matched households $h'$ iterate their demand to their match, i.e. $D_{h'}^t = \{\varphi_A^{t-1}(h')\}$.

For all $i \in I$ and $a \in A$, we define the set of households of type $i$ that demand apartment $a$ as follows:

$$H_{a,i}^t = \{h \in H \mid D_h^t = \{a\} \text{ and } i = \tau(h)\}.$$

The set of pairs $(a, h)$ that can be assigned to institution $i$ is defined as

$$M_i^t = \{(a, h) \in A \times H \mid (a, h) \succ^i \varnothing \text{ and } h \in H_{a,i}^t\}.$$

**$B^t$. Iteration over $\mathbf{M_i^t}$ to match institutions and apartments (the inner loop)**

Let $\theta^s(i) = \varnothing$ for all $i \in I$, and $\tilde{M}_i^s = M_i^t$, $s := 1$.

**B$^t$.1** All institutions $i$ demand the set of pairs $Ch_i(\tilde{M}_i^s, q_i)$. So, the set of institutions that demand an apartment $a$ is

$$I_a^s = \{i \in I \mid \text{ there exists } (a,h) \in Ch_i(\tilde{M}_i^s, q_i)\}.$$

**B$^t$.2** For all apartments $a$ such that $I_a^s \neq \varnothing$, apartment $a$ is assigned to the institution with the highest priority under $\pi_a$, i.e. $a \in \theta^s(i)$ if and only if $i = \max_{\pi_a} I_a^s$.

For all institutions $i$, let $\tilde{M}_i^{s+1} := \tilde{M}_i^s \setminus \{(a,h) \in \tilde{M}_i^s \mid (a,h) \in Ch_i(\tilde{M}_i^s, q_i) \text{ and } a \notin \theta^s(i)\}$. That is to say, we delete from the set $\tilde{M}_i^s$ those pairs where the institution $i$ is rejected.

If $|\theta^s(i)| = q_i$ for all institutions $i$, or $\tilde{M}_i^{s+1} = \varnothing$ for all institutions $i$ for which $|\theta^s(i)| < q_i$, go to B$^t$.3; otherwise, let $s = s+1$ and go to B$^t$.1.

**B$^t$.3** Rename $\theta^t(i) := \theta^S(i)$, where $S$ is the last iteration of B$^t$.1. Furthermore, each pair $(a,i)$ is tentatively assigned to household $h$ if and only if $a \in \theta^t(i)$ and $(a,h) \in Ch_i(\tilde{M}_i^S, q_i)$. That is to say $\varphi^t(h) = (a,i)$.

**C$^t$. Iteration over $\mathbf{D_h^t}$** For all unassigned households $h$, let $A_h^{t+1} := A_h^t \setminus \{\max_{P_h} A_h^t\}$. If each household has been rejected by all the apartments in her preference list or is matched, the tentative assignment becomes the output assignment. Otherwise, $t := t+1$, go to A$^t$.

## Appendix B. Examples Details

**Details of example 5.1:**

Running the NDA algorithm, the elicited demand of households, at step 1, is described in Table B.1. Also, Table B.2 shows the institutions that demand each apartment, where the apartment $a_1$ is demanded by both institutions.

| $I$ | $H^1_{a_1,i}$ | $H^1_{a_2,i}$ | $H^1_{a_3,i}$ | $M_i^1 = \tilde{M}_i^1$ |
|---|---|---|---|---|
| 1 | $h_1, h_6$ | $h_2, h_5$ | $\varnothing$ | $(a_1,h_1),(a_2,h_2)$ |
| 2 | $h_3, h_4, h_7$ | $\varnothing$ | $\varnothing$ | $(a_1,h_3),(a_1,h_4),(a_1,h_7)$ |

Table B.1: A$^1$. Elicited Demand of Households.

Note that $2\pi_{a_1}1$, i.e., Phase B$^1$ stops in one step. Consequently, the tentative assignment produced at the end of step 1 is given in the last column of Table B.3.

| $A$ | $I_a^1$ |
|---|---|
| $a_1$ | $1,2$ |
| $a_2$ | $2$ |
| $a_3$ | $\varnothing$ |

Table B.2: B$^1$. Institutions demand.

| $I$ | $Ch_i^1(\tilde{M}_i^1, q_i)$ | $\mu^1$ |
|---|---|---|
| 1 | $(a_1,h_1),(a_2,h_2)$ | $(a_2,h_2)$ |
| 2 | $(a_1,h_3)$ | $(a_1,h_3)$ |

Table B.3: B$^1$. Iteration over the sets $M_i^1$.

| $I$ | $H^2_{a_1,i}$ | $H^2_{a_2,i}$ | $H^2_{a_3,i}$ | $M^2_i = \tilde{M}^1_i$ |
|---|---|---|---|---|
| 1 | $h_5$ | $h_2, h_6, h_1$ | $\varnothing$ | $(a_2, h_1), (a_2, h_2)$ |
| 2 | $h_3$ | $h_4, h_7$ | $\varnothing$ | $(a_1, h_3), (a_2, h_4), (a_2, h_7)$ |

Table B.4: A$^2$. Elicited Demand of Households.

The algorithm continues to step 2, where Table B.4 shows the demands of households at this step, and Table B.5 summarizes the institutions that demand each apartment. Notice that institution 2 has a higher priority than institution 1 under priorities $\pi_{a_1}$ and $\pi_{a_2}$. Moreover, we have that $q_2 = 1$ and $(a_2, h_4) \succ^i (a, h)$ for all $(a, h) \in M^2_2$. So, the tentative assignment produced at the end of step 2 is $\mu^2$, which is shown in the last column of Table B.6.

| $A$ | $I^2_a$ |
|---|---|
| $a_1$ | $1, 2$ |
| $a_2$ | $1, 2$ |
| $a_3$ | $\varnothing$ |

Table B.5: B$^2$. Elicited demand of institutions.

| $I$ | $Ch^2_i(\tilde{M}^1_i, q_i)$ | $\mu^2$ |
|---|---|---|
| 1 | $(a_2, h_1)$ | $\varnothing$ |
| 2 | $(a_2, h_4)$ | $(a_2, h_4)$ |

Table B.6: B$^2$. Iteration over the sets $M^2_i$.

It is important to note that the assignments $(h_3, a_1, 2)$ and $(h_2, a_2, 1)$ are disrupted at the end of step 2 because $2\pi_{a_1}1$ and $(a_2, h_4) \succ^2 (a_1, h_3)$. Households $h_2$ and $h_3$ were rejected from apartments $a_2$ and $a_1$, respectively.

The algorithm continues to step 3 because not all households have been rejected by all their acceptable apartments. For example households $h_2$ and $h_3$ have not been rejected from the apartments $a_1$ and $a_2$, respectively. The demand of households at step 3 is shown in Table B.7.

| $I$ | $H^3_{a_1,i}$ | $H^3_{a_2,i}$ | $H^3_{a_3,i}$ | $M^3_i = \tilde{M}^1_i$ |
|---|---|---|---|---|
| 1 | $h_2$ | $\varnothing$ | $h_1, h_5, h_6$ | $(a_1, h_2), (a_3, h_1), (a_3, h_6)$ |
| 2 | $\varnothing$ | $h_3, h_4$ | $h_7$ | $(a_2, h_3), (a_2, h_4), (a_3, h_7)$ |

Table B.7: A$^3$. Elicited Demand of Households.

By Table B.7, we note that each apartment is demanded by a different institution. So, the tentative assignment produced at the end of step 3 is shown in the last column of Table B.8.

At the end of step 3, we note that $h_5$, $h_6$ and $h_7$ have been rejected from all their acceptable apartments. However, $h_4$ is rejected by $a_2$, and her last acceptable apartment is $a_3$. Thus, the algorithm continues to step 4, where $h_4$ demands the apartment $a_3$, and households $h_1$, $h_2$, $h_3$ iterate their demand to their match. Consequently, institutions 1 and 2 demand the apartment $a_3$ due to $(a_3, h_1) \succ^1 \varnothing$ and

| $I$ | $Ch_i^3(\tilde{M}_i^1, q_i)$ | $\mu^3$ |
|---|---|---|
| 1 | $(a_1, h_2), (a_3, h_1)$ | $(a_1, h_2), (a_3, h_1)$ |
| 2 | $(a_2, h_3)$ | $(a_2, h_3)$ |

Table B.8: B$^3$. Iteration over the sets $M_i^3$.

$(a_3, h_4) \succ \varnothing$. However, we know that institution 1 has a higher priority than institution 2 under the priority $\pi_{a_3}$, which implies that household $h_4$ is rejected from the apartment $a_3$ because $\tau(4) = 2$.

Therefore, the NDA algorithm stops at the end of step 4, and produces the assignment

$$\mu^{NDA} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_3 & a_1 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & 1 & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

**Details of example 5.2:**

Given that Example 5.1 corresponds to stage 0.1 of the NDAI, we delete all the pairs $(a_1, h)$ from the priority $\succ^2$ at Stage 0.2. We get

$$\succ^{1,1} = \succ^1 \text{ and } \succ^{2,1} = \begin{pmatrix} (a_2, h_3) \\ (a_2, h_4) \\ (a_2, h_7) \end{pmatrix}.$$

| $I$ | $H_{a_1}^{i1}$ | $H_{a_2}^{i1}$ | $H_{a_3}^{i1}$ |
|---|---|---|---|
| 1 | $h_1, h_6$ | $h_2, h_5$ | $\varnothing$ |
| 2 | $h_3,\ h_4, h_7$ | $\varnothing$ | $\varnothing$ |

Table B.9: A$^1$. Elicited Demand of Households, Stage 1.1

Consider stage 1.1. We run the NDA algorithm with priorities $\succ^1$. Step 1.1 of this NDA algorithm is summarized in Table B.9. Given that $q_1 = 2$, and $1\pi_{a_1}2$, the tentative assignment is

$$\mu^1 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & 1 & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

The NDA in Stage 1.1 moves to step 2. Phase A$^2$ of the algorithm is illustrated in the Table B.10. Following the institutions' priorities and the fact that $2\pi_{a_3}1$, the tentative assignment produced at the end of step 2 is

$$\mu^2 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & \varnothing & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & \varnothing & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

| $I$ | $H^2_{a_1,i}$ | $H^2_{a_2,i}$ | $H^2_{a_3,i}$ |
|---|---|---|---|
| 1 | $h_1, h_5$ | $h_2, h_6$ | $\varnothing$ |
| 2 | $\varnothing$ | $h_3, h_4, h_7$ | $\varnothing$ |

Table B.10: A$^2$. Elicited Demand of Households.

Now, the NDA algorithm of Stage 1.1 moves to step 3. We show the demands of households in Table B.11. Since the institution 1 has a higher priority than institution 2 under $\pi_{a_3}$, we get the tentative assignment $\mu^3$.

$$\mu^3 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & \varnothing & a_2 & \varnothing & a_3 & \varnothing & \varnothing \\ 1 & \varnothing & 2 & \varnothing & 1 & \varnothing & \varnothing \end{pmatrix}.$$

| $I$ | $H^3_{a_1,i}$ | $H^3_{a_2,i}$ | $H^3_{a_3,i}$ |
|---|---|---|---|
| 1 | $h_1, h_2$ | $\varnothing$ | $h_5, h_6$ |
| 2 | $\varnothing$ | $h_3$ | $h_4, h_7$ |

Table B.11: A$^3$. Elicited Demand of Households.

Note that household $h_2$ has not been rejected by the apartment $a_3$, her last acceptable apartment. The NDA at Stage 1.1 moves to step 4 where Table B.12 shows Phase A$^4$.

| $I$ | $H^4_{a_1,i}$ | $H^4_{a_2,i}$ | $H^4_{a_3,i}$ |
|---|---|---|---|
| 1 | $h_1$ | $\varnothing$ | $h_2, h_6$ |
| 2 | $\varnothing$ | $h_3$ | $\varnothing$ |

Table B.12: A$^4$. Elicited Demand of Households.

We observe that all households have been accepted or rejected at the end of the step 4, and hence Stage 1.1 stops. There are no interrupters because no apartment is rejected by any institution. Therefore, the NDAI algorithm stops and produces the following assignment.

$$\mu^{NDAI} = \mu^4 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ a_1 & a_3 & a_2 & \varnothing & \varnothing & \varnothing & \varnothing \\ 1 & 1 & 2 & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

□

## Appendix C. Proofs

**Proof of Lemma 3.1:**

Consider a step $t$ of the NDA algorithm. We know that phase $\mathrm{B}^t$ starts with the set of acceptable apartment-households pairs $\tilde{M}_i^1 = M_i^t$ for all institutions $i$. This implies that each institution $i$ demands the set $Ch_i(\tilde{M}_i^t, q_i)$, a set where apartments and households are not paired twice.

Let $A_i^t = \{a \in A | (a,h) \in M_i^t\}$ be the set of apartments demanded by some household of type $i$ at step $t$. Since $|A_i^t| \leq A$, there are no distributional constraints and priorities are responsive, for all apartments $a$ all top acceptable pairs $(a,h)$ belong to the set $Ch_i(\tilde{M}_i^1, q_i)$. Each institution $i$ demands all the apartments in the set $A_i^t$. So, all the apartments in $\bigcup_{i \in I} A_i^t$ belong to $\theta^1(j)$, for some $j \in I$, at the end of Phase $\mathrm{B}^t$.2.

Consequently

$$\tilde{M}_i^2 = \tilde{M}_i^1 \setminus \{(a,h) \in \tilde{M}_i^1 | a \in \theta^1(j) \text{ for some } j \neq i\} = \tilde{M}_i^1 \setminus \tilde{M}_i^1 = \varnothing$$

for all institution $i$. Therefore, phase $\mathrm{B}^t$ stops in one iteration.

**Proof of Lemma 3.2:** Consider an apartment $a$ that is assigned by some institution $i$ at some step $t$, i.e. $a$ belongs to $\theta^t(i)$. Since households iterate their demand to their match, we have that $i \in I_a^{t+1}$. In other words, this apartment is demanded by some institution at step $t+1$. Since there are no distributional constraints and priorities are responsive, the apartment $a$ is assigned to some institution at the end of step $t+1$ (the institution in $I_a^{t+1}$ with the highest priority at $\pi_a$). Iterating this argument, we conclude that the apartment $a$ is assigned to some institution at all steps $t' \geq t$.

Therefore $\mu^{NDA}(a) \neq \varnothing$ because the NDA algorithm stops in a finite number of steps.

**Proof of Theorem 3.1:**

**Individual Rationality**. For all institutions $i \in I$, we know that $\mu^{NDA}(i) \subseteq Ch_i(M_i^T, q_i)$, where $T$ is the last iteration of the NDA algorithm. Thus, $(a,h) \succ^i \varnothing$ for all $(a,h) \in \mu^{NDA}(i)$. Therefore, $\mu^{NDA}(i) \succ \varnothing$ for all $i \in I$.

Moreover, the NDA algorithm stops when every unmatched household has been rejected by all her acceptable apartments, in this case $\varphi(h) = \varnothing$, or every household is matched to some acceptable apartment, i.e. $\varphi_A(h) P_h \varnothing$ for all $h \in H$.

**Non-wastefulness.** We proceed by contradiction. We assume the existence of a household-institution pair $(h,i)$ that claims an empty apartment $a$. That is to say, we have that i) $a P_h \varphi_A(h)$, ii) $(a,h) \in Ch_i(\mu^{NDA}(i) \cup \{(a,h)\}, q_i)$, and iii) $\theta^{NDA}(a) = \varnothing$.

The condition i) means that household $h$ demands the apartment $a$ at some step $t$ of the NDA algorithm. Moreover, condition ii) guarantees that the pair $(a,h)$ is acceptable for the institution $i$. Thus, institution $i$ demands the apartment $a$ at step $t$. Applying Lemma 3.2, there exists some institution $j$ such that $a \in \theta^{NDA}(j)$ which contradicts the condition iii).

**There is no justified envy**. Suppose, on the contrary, the existence of a pair $(h,i)$ that has justified envy over a pair $(h', i')$, where $\tau(h) = i$ and $\tau(h') = i'$. Then, there exists an apartment $a$ such that $\varphi_A^{NDA}(h') = a$, $a \in \theta(i')$, and

**i.** $aP_h\varphi_A^{NDA}(h)$,

**ii.** $(a,h) \in Ch_i(\mu^{NDA}(i) \cup \{(a,h)\}, q_i)$,

**iii.** $i\pi_a i'$.

By condition i), household $h$ demands the apartment $a$ at some step $t$. Moreover, condition ii) ensures that the pair $(a,h)$ is acceptable for the institution $i$, i.e. $(a,h) \in M_i^t$. Consequently, we have that $i \in I_a^t$. We analyze the following cases.

**Case 1.** $i = i'$, i.e. $\tau(h) = \tau(h') = i$. Since $\varphi_A^{NDA}(h') = a$, household $h'$ demands the apartment $a$ at some step $t'$. Even more, $(a,h') \in Ch_i(M_i^{t'}, q_i)$ because $(a,h')$ belongs to the set $\mu^{NDA}(i)$. Given that $(a,h) \in M_i^t$ but $\varphi_A^{NDA}(h) \neq a$, the responsiveness of priorities ensure that

$$\{(a,h')\} \succ^i \{(a,h)\}. \tag{C.1}$$

Moreover, since no apartment can be paired twice, condition ii) implies that

$$\{(a,h)\} \succ^i \{(a,h')\},$$

in contradiction with (C.1).

**Case 2.** $i \neq i'$, i.e. $\tau(h) \neq \tau(h')$. We know that $a \in \theta^{NDA}(i')$, which implies the existence of some step $t'$ where

$$i'\pi_a j \text{ for all } j \in I_a^t, \text{ for all } t \geq t',$$

according to Phase B$^t$.3. In particular

$$i'\pi_a i,$$

in contradiction with condition iii).

In all cases we get a contradiction, therefore there is no justified envy at the assignment $\mu^{NDA}$. So, this assignment is fair.

**Pareto undominated**. We proceed as in Gale and Shapley [47]. To prove that $\mu^{NDA}$ is Pareto undominated, we show that in any other fair assignment, each household gets the same apartment or an apartment less preferred than $\varphi^{NDA}(h)$.

An apartment $a$ is said to be **achievable** for a household $h$ if there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi_A^\mu(h) = a$. We proceed by induction to show that no household is rejected by an achievable apartment during the NDA algorithm.

*Hypothesis of induction.* At step $t$ we assume that no household has been rejected by an achievable apartment. In other words, if a household is rejected by some apartment, this apartment is not achievable for her.

*Induction step.* Consider that some household $h^*$ is rejected at step $t+1$ from an apartment $a$. We assume, on the contrary, that $a$ is achievable for household $h^*$. Thus, there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu(h^*) = (a, i^*)$. So, the pair $(a, h^*)$ is acceptable for the institution $i^*$.

Now, let $h$ be the household assigned to the apartment $a$ at the end of step $t+1$, this means that $\varphi^{t+1}(h) = (a,i)$ where $i = \tau(h)$. We analyze the following cases.

**Case 1.** $i = i^*$. Since $\varphi^{t+1}(h) = (a,i)$, the apartment $a$ belongs to $\theta^{t+1}(i)$. That is to say

$$\{(a,h)\} \succ^i \{(a,h^*)\} \tag{C.2}$$

because $(a,h^*) \notin \mu^{t+1}(i)$. Since priorities are responsive, we have that

$$(a,h) \in Ch_i(\mu(i) \cup \{(a,h)\}).$$

Note that $h$ prefers $a$ to all the apartments that have not rejected her, then the induction hypothesis ensures that household $h$ prefers $a$ to any other achievable apartment for her:

$$aP_h\varphi^{\mu}_A(h).$$

Moreover, $(a,h^*) \in \mu(i)$. That is to say, the pair $(h,i)$ has justified envy over the pair $(h^*,i)$ at the apartment $a$ in the assignment $\mu$, which contradicts the fact that $\mu$ is a fair assignment.

**Case 2.** $i \neq i^*$. We know that $\varphi^{\mu}(h^*) = (a,i^*)$, i.e. the pair $(a,h^*)$ is acceptable for the institution $i^*$, so $i^* \in I^{t+1}_a$. Moreover, $i \in I^{t+1}_a$ because $\varphi^{t+1}(h) = (a,i)$. Given that $a \in \theta^{t+1}(i)$, we conclude that $a \notin \theta^{t+1}(i^*)$ because $i = \max_{\pi_a} I^{t+1}_a$; thus

$$i\pi_a i^*. \tag{C.3}$$

By the induction hypothesis, we know that household $h$ strictly prefers $a$ to any other achievable apartment for her, i.e.

$$aP_h\varphi^{\mu}_A(h). \tag{C.4}$$

Now, we know that $(a,h) \in \mu^{t+1}(i)$, this means that the pair $(a,h)$ is acceptable for institution $i$. Moreover, $(a,h^*) \in \theta^{\mu}(i^*)$, thus $a$ is not assigned to $i$ at $\mu$. Also, there are no distributional constraints and institutions' priorities are responsive, then

$$(a,h) \in Ch_i(\mu(i) \cup \{(a,h)\}, q_i). \tag{C.5}$$

By C.3, C.4 and C.5, the pair $(h,i)$ has justified envy over the pair $(h^*,i^*)$ at the apartment $a$ in the assignment $\mu$, which contradicts the fact that $\mu$ is fair.

In any case, a contradiction arises when we assume that household $h^*$ is rejected by some achievable apartment $a$. So, no household is rejected by an achievable apartment. Therefore, $\mu^{NDA}$ is Pareto undominated by fair assignments.

**Truth-telling is a dominant strategy for households**. We construct the proof as in Roth [48].

For each household $h$, we say that $P'_h$ is a **successful** misrepresentation of $P_h$ if $P'_h$ is a preference list such that

$$\varphi^{NDA}_A[P'_h, P_{-h}](h)P_h\varphi_A[P](h).$$

Let $a' := \varphi^{NDA}_A[P'_h, P_{-h}](h)$, we define the preference list $P''_h$ where the apartment $a'$ is declared as the most preferred apartment of $h$. Let $P'$ and $P''$ be the preference profiles where household $h$ reports $P'_h$

and $P''_h$, respectively, and other households do not change their true preferences. The following lemma establishes that $P'_h$ and $P''_h$ are equivalents in the sense that

$$\varphi_A^{NDA}[P'_h, P_{-h}](h) = \varphi_A^{NDA}[P''_h, P_{-h}](h).$$

**Lemma Appendix C.1.** *Consider a matching market through institutions with no distributional constraints. Then* $\varphi_A^{NDA}[P'_h, P_{-h}](h) = \varphi_A^{NDA}[P''_h, P_{-h}](h)$.

*Proof.* By paragraphs above, the assignment $\mu^{NDA}[P']$ is fair with respect to $P'$. Let $i := \tau(h)$, it is no possible at $\varphi_A^{NDA}[P'_h, P_h]$ that $(h, i)$ has justified envy over other pairs under $P''$ because no apartment is preferred to $a'$ under $P''_h$. Since other preferences did not change between $P'$ and $P''$, it implies 1. $\mu^{NDA}[P']$ is fair with respect to the profile $P''$, i.e. $a'$ is achievable for $h$ under the preference profile $P''$; 2. the apartment $a'$ is the best acceptable achievable apartment of $h$ under $P''$, and since $\mu^{NDA}[P'']$ is Pareto undominated by other fair assignments, we conclude that

$$\mu^{NDA}[P'](h) = \mu^{NDA}[P''](h).$$

□

The following Lemma establishes that households are not worse off when a household successfully misrepresents her true preference list.

**Lemma Appendix C.2.** *Consider $P'_h$ a preference list different from the true preference list of $h$. If $\varphi_A^{NDA}[P'](h)$ is weakly preferred to $\varphi_A^{NDA}[P](h)$, then for each household $h' \neq h$, either*

$$\varphi_A^{NDA}[P'](h') P_{h'} \varphi_A^{NDA}[P](h') \text{ or } \varphi_A^{NDA}[P'](h') = \varphi_A^{NDA}[P](h').$$

*Proof.* We proceed by contradiction, i.e. we assume that $aP_{h'}a'$ for some $h' \in H$, where

$$\varphi_A^{NDA}[P'](h') = a' \text{ and } \varphi_A^{NDA}[P](h') = a \text{ for } h' \neq h.$$

So, there exists a step $t$ of $NDA[P']$ at which $h'$ is rejected from $a$. Hence, by Lemma 3.2, the apartment $a$ is assigned to some household $h''$. We analyze the following cases.

**Case 1.** If $\tau(h'') = \tau(h') = i$. Since $\mu^{NDA}[P']$ is fair, we have that $\{(a, h'')\} \succ^i \{(a, h')\}$. Let $a'' := \varphi_A^{NDA}[P](h'')$, we have the following sub-cases.

**Case 1.1** $aP_{h''}a''$. Then $(h'', i)$ has justified envy over $(h, i)$ at apartment $a$ in the assignment $\mu^{NDA}[P]$, a contradiction.

**Case 1.2** $a''P_{h''}a$. By Lemma 3.2, there exists a household $h'''$ such that $\varphi_A^{NDA}[P'](h''') = a''$. So, we apply the previous reasoning on $h''$ to $h'''$, which will end up generating either a contradiction, or an infinite succession of households $\{h^{(k)}\}$. This is not possible because $H$ is finite.

**Case 2.** If $i = \tau(h') \neq \tau(h'') = i'$. Since $\mu^{NDA}[P']$ is fair, we have that $i'\pi_a i$. Let $a'' := \varphi_A^{NDA}[P](h'')$, we have the following sub-cases.

**Case 2.1** $aP_{h''}a''$. Then $(h'', i')$ has justified envy over $(h, i)$ at apartment $a$ in the assignment $\mu^{NDA}[P]$, a contradiction.

**Case 2.2** $a''P_{h''}a$. By Lemma 3.2, there exists a household $h'''$ such that $\varphi_A^{NDA}[P'](h''') = a''$. So, we apply the previous reasoning on $h''$ to $h'''$, which will end up generating either a contradiction, or an infinite succession of households $\{h^{(k)}\}$. This is not possible because $H$ is finite.

Therefore, no household $h'$ is worse off under the assignment $\mu^{NDA}[P']$. □

We are ready to prove that truth-telling is a dominant strategy for all households. We assume, on the contrary, the existence of household $h^*$ and a successful misrepresentation $P'_{h^*}$ of $P_{h^*}$. That is to say

$$a' P_{h^*} a,$$

where $\varphi_A^{NDA}[P](h^*) = a$ and $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}] = a'$.

By Lemma Appendix C.1, we consider that $P_{h^*}$ is the preference list where $a'$ is the most preferred apartment. Our objective is to show that $P'_{h^*}$ is not a successful manipulation. To do that, we follow the proof of Roth [48] about the strategy-proofness of the DA. We say that household $h$ **makes a match** at step $t$ of the NDA algorithm, if $h$ demands $\varphi_A^{NDA}(h)$ at step $t$. This proof analyzes the possible steps where $h^*$ makes a match.

**Lemma Appendix C.3.** *Consider the household $h^*$ with preferences $P_{h^*}$ and $P'_{h^*}$ such that*

$$\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h^*) R_{h^*} \varphi_A^{NDA}[P](h^*).$$

*If $\varphi_A^{NDA}[P](h^0) = \varnothing$ then $\varphi_A^{NDA}[P'](h^0) = \varnothing$.*

*Proof.* By contradiction, suppose that $h^0$ gets an apartment at $P'$. Since assignments $\mu^{NDA}[P]$ and $\mu^{NDA}[P']$ are non-wasteful, this means that some household $h$, previously matched at $P$, is unmatched at $P'$, violating Lemma Appendix C.2. □

First, if a household makes a match in the last step of the NDA algorithm, under the true preferences, then no manipulation is a successful misrepresentation of her true preference list.

**Claim Appendix C.1.** Suppose that $h^*$ makes a match at $t^*$, with $1 \leq t^* \leq T$, then $\varphi_A^{NDA}[P'](h) = \varphi_A^{NDA}[P](h)$ for all $h$ that makes a match at $T$. Moreover, if $h^*$ makes a match at $T$, there is no profitable deviation $P'_{h*}$ of her true preference list $P_{h^*}$.

*Proof.* First, we present the argument for $h$. Let $T$ be the last step of the $NDA[P]$ and consider that the household $h$ makes a match at step $T$, say $a = \varphi_A^{NDA}[P](h)$. Since $\mu^{NDA}[P]$ is non-wasteful, all apartments are matched, at $T-1$ either

1. $a$ is unmatched, or
2. $a$ is matched to a household $h_1$ who is unmatched at $\mu^{NDA}[P]$.

**Case 1**. Since apartment $a$ was unmatched at $T-1$, all matched households prefer their match at $\mu^{NDA}[P]$ to $a$. By Lemma Appendix C.2, this implies that none of them gets $a$ at $\mu^{NDA}[P'_{h^*}, P_{-h^*}]$. By Lemma Appendix C.3 all unmatched households are still unmatched. So, by non-wastefulness, $h$ gets $a$ and does not strictly improve her match under the profile $(P'_{h^*}, P_{-h^*})$.

**Case 2**. Let $h_1$ be the household matched with $a$ at $T-1$, who is unmatched at $T$. For all matched $h_2 \neq h_1$, if any, who prefer $a$ to $\varphi_A^{NDA}[P](h_2)$, we have that $\{(a,h_1)\} \succsim_{\tau(h_1)} \{(a,h_2)\}$, by fairness of $\mu^{NDA}[P]$, or $\tau(h_1)\pi_a\tau(h_2)$. Thus, if $h$ strictly improves under $(P'_{h^*}, P_{-h^*})$, then $h_1$, or an unmatched household, gets $a$ or an apartment preferred to $a$ at $\mu^{NDA}[P'_{h^*}, P_{-h^*}]$ because the assignment is fair; in contradiction to Lemma Appendix C.3.

In both cases we conclude that $\varphi_A^{NDA}[P'_{h^*}, P_{h^*}](h) = \varphi_A[P](h)$.

Second, the argument is the same for $h^*$ that makes a match at $T$ since other households $h$ do not improve her allocation under $(P'_{h^*}, P_{-h^*})$ (Lemma A.2). □

Now, we consider that $h^*$ makes a match at some step $t$ of the $NDA[P]$ procedure, with $1 \leq t < T$. We show that no household, matched after $t$, changes its final allocation when household $h^*$ misrepresents her true preference list through $P'_{h^*}$.

**Claim Appendix C.2.** Suppose that $h^*$ makes a match at $t^*$, $1 \leq t^* \leq T$ and $P'_{h^*}$ and $P_{h^*}$ are such that

$$\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h^*) R_{h^*} \varphi_A^{NDA}[P](h^*).$$

Then $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h^t) = \varphi_A^{NDA}[P](h^t)$ for $h^t \neq h^*$ who makes a match at $t$, where $t^* \leq t \leq T$.

*Proof.* The proof is by induction.

**Base of Induction**. Starts in $t = T$. It is true by Claim Appendix C.1.

**Hypothesis of Induction**. Suppose the property is true until step $t+1$.

**Induction Step**. Let $a^t$ be the match of $h^t$ at $\varphi_A^{NDA}[P](h^t)$.

**Case 1**. $a^t$ is unmatched at $t-1$. Since $a^t$ is unmatched at $t-1$, all households matched before/at $t$ strictly prefer their match to $a^t$, by Lemma Appendix C.2 they do not get $a^t$ at $P'$. By induction hypothesis and Lemma Appendix C.3, those who make a match after $t$ get the same apartment or nothing, i.e. non-wastefulness of $\mu^{NDA}$ guarantees that $\varphi_A^{NDA}[P'_{h^*}, P_{-h^*}](h^{t-1}) = \varphi_A^{NDA}[P](h^{t-1})$.

**Case 2**. $a^t$ is matched at $t-1$. Let $h^{t-1}$ be the match of $a^t$ at $t-1$; thus $h^{t-1}$ has top priority among households who prefer $a^t$ to their match and make a match before/at $t$. By Lemma Appendix C.3, fairness of $\varphi_A^{NDA}[P']$ and Lemma Appendix C.2, $h^{t-1}$ or a household $h'$ that makes a match after $t$ should get $a^t$, or an apartment prefer to $a^t$, at $\varphi_A^{NDA}[P']$. Since $h^{t-1}$ and $h'$ makes a match, if any, after $t$, it is not the case by the hypothesis of induction. □

In any step where household $h^*$ makes a match, Claim Appendix C.2 implies that $\varphi_A^{NDA}[P'](h^*) = \varphi_A^{NDA}[P](h^*)$. That is to say, there is no successful misrepresentation of $P_{h^*}$, and the NDA is strategy-proof.

**Proof of Theorem 5.1** : Let $x^*$ be the last iteration of the NDAI mechanism, i.e. there are no interrupter at the end of the NDA phase $x^*.1$. The last step of the NDA phase is denoted by $T$.

**Individual Rationality**. For all institutions $i \in I$, we know that $\mu^{NDAI}(i) \subseteq Ch_i(M_i^T, q_i)$, where $T$ is the last iteration of the NDA phase. Thus, $(a,h) \succ^i \varnothing$ for all $(a,h) \in \mu^{NDAI}$. Therefore, $\mu^{NDAI}(i) \succ \varnothing$ for all $i \in I$. Moreover, the NDA algorithm stops when every unmatched household has been rejected by all her acceptable apartments, in this case $\varphi(h) = \varnothing$, or every household is matched to some acceptable apartment, i.e. $\varphi_A(h) P_h \varnothing$ for all $h \in H$.

**Distributional Constraints**. Consider an institution $i$ that does not fulfill its quota in the assignment $\mu^{NDAI}$. Thus, $|\mu^{NDAI}(i)| < q_i$, i.e. there is at least one apartment $a$ that remains unassigned. We know that $\mu^{NDAI}$ is IR, and the over-demand condition holds in the market $E$, then for all institutions $i$ and, particularly, the apartment $a$, there exists an unassigned household $h$ such that $(a,h)$ is acceptable for $i$ and $aP_h\varnothing$. Considering the NDA phase at Stage $x^*.1$ we have that $(a,h)$ belongs to $M_i^t$, for some step $t$, or not.

**Case 1**. If $(a,h) \in M_i^t$, since $i$ did not fulfill its quota and $(a,h)$ is acceptable for $i$ whose priorities are responsive, it means that $(a,h)$ has not been assigned to $i$; this implies the existence of an institution $j$ such that $j\pi_a i$, i.e. $a \in \theta(j)^{t'}$ for some step $t' \geq t$. However, we know that $a \notin \mu^{NDAI}(j)$. Hence, the institution $j$ is an interrupter for $a$, which contradicts the fact that there are no interrupters at $x^*$.

**Case 2**. $(a,h) \notin M_i^t$ because of $\succ^{i,x^*}$, i.e. the priority of institution $i$ at stage $x^*$ since the NDAI algorithm actualizes institutions' priorities. This case only happens if $i$ is an interrupter over $a$ and fulfill its quota, thus $(a,h)$ is deleted from $\succ^{i,x}$ at some stage $x$ of the NDAI algorithm. This is not possible because the institutions did not fulfill its quota.

In any case we get a contradiction. Therefore, we have that $|\mu^{NDAI}(i)| = q_i$ for all $i \in I$.

**Non-wastefulness.** Follows from the fact that each institution fills its quota.

**There is no justified envy.** Let $x^*$ be the last iteration of the NDAI. Suppose, on the contrary, the existence of a pair $(h,i)$ that has justified envy over a pair $(h', i')$, where $\tau(h) = i$ and $\tau(h') = i'$. Then, there exists an apartment $a$ such that $\varphi_A^{NDAI}(h') = a$, $a \in \theta(i')$, and

**i.** $aP_h\varphi_A^{NDAI}(h)$,

**ii.** $(a,h) \in Ch_i(\mu^{NDA}(i) \cup \{(a,h)\}, q_i)$,

**iii.** $i\pi_a i'$.

By condition i), household $h$ demands the apartment $a$ at some step $t$. Moreover, the condition ii) ensures that the pair $(a,h)$ is acceptable for the institution $i$, i.e. $(a,h) \in M_i^t$. Consequently, we have that $i \in I_a^t$. We analyze the following cases.

**Case 1.** $i = i'$, i.e. $\tau(h) = \tau(h') = i$. Since $\varphi_A^{NDAI}(h') = a$, household $h'$ demands the apartment $a$ at some step $t'$. Moreover condition 1 ensures that $h$ also demands the apartment $a$ at some step $t'$, by responsiveness of priorities at each step the institution $i$ picks the top pair with $a$, thus

$$\{(a, h')\} \succ^i \{(a, h)\}. \tag{C.6}$$

Moreover, since no apartment can be paired twice and the fact that priorities are responsive, the condition ii) implies that $\{(a, h)\} \succ^i \{(a, h')\}$ in contradiction with (C.6).

The following Lemma is required to prove Case 2.

**Lemma Appendix C.4.** *Consider an institution $i$ with priorities $\succ^i$ and suppose that $|\tau(h)| = 1$ for all $h \in H$. If $i$ is an interrupter over an apartment $a$ through a household $h$, then $(h, i)$ has not justified envy over other pairs $(h^a, i^a)$ at $\mu^{NDAI}$, where $(a, h^a) \in \mu^{NDAI}(i^a)$.*

**Proof of Lemma Appendix C.4**. Since $i$ is an interrupter, there exist steps $\underline{t}$ and $\bar{t}$ such that $(a, h) \in \mu^t(i)$ for all $t \in [\underline{t}, \bar{t}]$, but $(a, h) \notin \mu^{t'}(i)$ for all $t' \geq \bar{t}$.

Because priorities of institutions are responsive, at $\bar{t} + 1$ institution $i$ drops $(h, a)$ only if it has filled its quota and $\{(h^{\bar{t}+1}, a^{\bar{t}+1})\} \succ^i \{(h, a)\}$ for all $(h^{\bar{t}+1}, a^{\bar{t}+1}) \in \mu^{\bar{t}+1}(i)$.

Moreover, institutions only improve all along the sequence of tentative matchings, thus $(\overline{h}, \overline{a}) \succsim^i (\overline{h}^{\bar{t}+1}, a^{\bar{t}+1})$ for all $(\overline{h}, \overline{a}) \in \mu^{NDAI}(i)$ and $(\overline{h}^{\bar{t}+1}, a^{\bar{t}+1}) \in \mu^{\bar{t}+1}(i)$, and fills its quota at $\mu^{NDAI}$, so $(h, a) \notin Ch_i(\mu^{NDAI}(i) \cup \{(h, a)\}, q_i)$, thus $(h, i)$ has not justified envy over $(h^a, i^a)$ at $\mu^{NDAI}$, where $(a, h^a) \in \mu^{NDAI}(i^a)$. □

**Case 2.** $i \neq i'$, i.e. $\tau(h) \neq \tau(h')$.

**Case 2.1** Consider that $i$ is not an interrupter, then $i$ demands apartment $a$ at iteration $x^*$. Also, we know that $a \in \theta^{NDAI}(i')$, then there exists a step $t'$ such that $i' \pi_a j$ for all $j \in I^t_a,$ for all $t \geq t'$, according to the Phase B.3. Since $a \notin \theta^{NDAI}(i)$ we have that $i' \pi_a i$, in contradiction with the condition iii).

**Case 2.1** If $i$ is an interrupter, assuming that $(h, i)$ has justified envy over $(h', i')$, through apartment $a$, contradicts Lemma Appendix C.4.

In any case we get a contradiction, therefore there is no justified envy at the assignment $\mu^{NDA}$. So, this assignment is fair.

**Pareto Undominated for fair assignments**. We proceed as in Gale and Shapley [47]. To prove that $\mu^{NDA}$ is Pareto undominated, we show that in any other fair assignment, each household gets the same apartment or an apartment less preferred than $\varphi^{NDA}(h)$.

An apartment $a$ is said to be **achievable** for a household $h$ if there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu_A(h) = a$. We proceed by induction to show that no household is rejected by an achievable apartment during the NDA algorithm at iteration $x^*$ where there are not interrupters.

*Hypothesis of induction.* At step $t$, we assume that no household has been rejected by an achievable apartment. In other words, if a household is rejected by some apartment, this apartment is not achievable for her.

*Induction step.* Consider that some household $h^*$ is rejected at step $t + 1$ from an apartment $a$. We assume, on the contrary, that $a$ is achievable for household $h^*$. Thus, there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu(h^*) = (a, i^*)$. So, the pair $(a, h^*)$ is acceptable for the institution $i^*$.

Now, let $h$ be the household assigned to the apartment $a$ at the end of step $t+1$, this means that $\varphi^{t+1}(h) = (a,i)$ where $i = \tau(h)$. We analyze the following cases.

**Case 1.** $i = i^*$. Since $\varphi^{t+1}(h) = (a,i)$, the apartment $a$ belongs to $\theta^{t+1}(i)$. That is to say

$$\{(a,h)\} \succ^i \{(a,h^*)\} \tag{C.7}$$

because $(a,h^*) \notin \mu^{t+1}(i)$. Since priorities are responsive, we have that $(a,h) \in Ch_i(\mu(i) \cup \{(a,h)\}, q_i)$. Note that $h$ prefers $a$ to all the apartments that have not rejected her, then the induction hypothesis ensures that household $h$ prefers $a$ to any other achievable apartment for her $aP_h\varphi^{\mu}_A(h)$.

Moreover, $(a,h^*) \in \mu(i)$. That is to say, the pair $(h,i)$ has justified envy over the pair $(h^*,i)$ at the apartment $a$ in the assignment $\mu$, which contradicts the fact that $\mu$ is a fair assignment.

**Case 2.** $i \neq i^*$. We know that $\varphi^{\mu}(h^*) = (a,i^*)$, i.e. the pair $(a,h^*)$ is acceptable for the institution $i^*$, so $i^* \in I_a^{t+1}$. Since there are not interrupters at iteration $x^*$, the offer of $i^*$ is rejected only if $i$ fulfills its quota. So, there exist $(a_{\hat{x}}, h_{\hat{x}}) \in \mu^{t+1}(i^*)$ such that $\{(a_{\hat{x}}, h_{\hat{x}})\} \succ^{i^*} \{(a,h^*)\}$ where $\hat{x} = 1,2,\ldots,q_{i^*}$. We know that $(a,h^*) \in \mu(i^*)$, i.e., some pair $(a_{\hat{x}}, h_{\hat{x}}) \notin \mu(i^*)$. By hypothesis of induction, this implies that $h_{\hat{x}}$ prefers $a_{\hat{x}}$ to any other achievable apartment for her, i.e.

$$a_{\hat{x}} P_{h_{\hat{x}}} \varphi^{\mu}_A(h_{\hat{x}}). \tag{C.8}$$

Since $(a_{\hat{x}}, h_{\hat{x}}) \succ^{i^*} \varnothing$, by expression (C.8), we conclude that the pair $(h_{\hat{x}}, i^*)$ has justified envy over the pair $(h^*, i^*)$ in the assignment $\mu$, which contradicts the fact that $\mu$ is fair.

In any case, a contradiction arises when we assume that household $h^*$ is rejected by some achievable apartment $a$. So, no household is rejected by an achievable apartment. Therefore, $\mu^{NDA}$ is Pareto undominated by fair assignments.

**Strategy-Proofness**. For each household $h$, we say that $P'_h$ is a **successful** misrepresentation of $P_h$ if $P'_h$ is a preference list such that $\varphi^{NDAI}_A[P'_h, P_{-h}](h) P_h \varphi^{NDAI}_A[P](h)$. Let $a' := \varphi^{NDAI}_A[P'_h, P_{-h}](h)$, we define the preference list $P''_h$ where the apartment $a'$ is declared as the most preferred apartment of $h$. Let $P'$ and $P''$ be the preference profiles where household $h$ reports $P'_h$ and $P''_h$, respectively, and other households do not change their true preferences.

Let $P'_h$ be a misrepresentation of $P_h$ where apartment $a$ is acceptable. We analyze the following cases.

**Case 1**. The institution $i$ is an interrupter for apartment $a$ at some iteration $x$. As a consequence, all pairs $(a,h)$ are deleted from the priority $\succ^{i,x}$, i.e., pairs $(a,h)$ are not acceptable at priority $\succ^{i,x'}$ for all $x' > x$. So, household $h$ is not assigned to apartment $a$ in stages $x'.1$, with $x' > x$. Therefore, $P'_h$ is not a successful misrepresentation of $P_h$.

**Case 2**. The institution $i$ is not an interrupter for apartment $a$. To prove that $P'_h$ is not a successful misrepresentation, first, we prove the following Lemma.

**Lemma Appendix C.5.** *Consider a matching market where the over-demand condition holds. If an apartment is assigned at some step $t$ by some institution, this apartment is assigned under the assignment $\mu^{NDAI}$.*

**Proof of Lemma Appendix C.5:** Suppose that $x^*$ is the last iteration of the NDAI where there are no interrupters. Consider an apartment $a$ that is assigned by some institution $i$ at some step $t$, during the NDA phase of stage $x^*$, i.e. $a$ belongs to $\theta^t(i)$. Since households iterate their demand to their match, we have that $i \in I_a^{t+1}$. In other words, this apartment is demanded by some institution at step $t+1$. Since the over-demand condition holds, there are no interrupters and priorities are responsive, the apartment $a$ is assigned to some institution at the end of step $t+1$ (the institution in $I_a^{t+1}$ with the highest priority at $\pi_a$). Iterating this argument, we conclude that the apartment $a$ is assigned to some institution at all steps $t' \geq t$. □

Therefore $\mu^{NDAI}(a) \neq \varnothing$ because the NDA algorithm stops in a finite number of steps. We construct the proof as in Roth [48]. The following lemma establishes that $P'_h$ and $P''_h$ are equivalents in the sense that $\varphi_A^{NDAI}[P'_h, P_{-h}](h) = \varphi_A^{NDAI}[P''_h, P_{-h}](h)$.

**Lemma Appendix C.6.** *Consider a matching market through institutions where the over-demand condition holds. Then* $\varphi_A^{NDAI}[P'_h, P_{-h}](h) = \varphi_A^{NDAI}[P''_h, P_{-h}](h)$.

*Proof.* By paragraphs above, the assignment $\mu^{NDAI}[P']$ is fair with respect to $P'$. Let $i := \tau(h)$, it is no possible at $\varphi_A^{NDAI}[P'_h, P_h]$ that $(h, i)$ has justified envy over other pairs under $P''$ because no apartment is preferred to $a'$ under $P''_h$. Since other preferences did not change between $P'$ and $P''$, it implies that: 1. $\mu^{NDAI}[P']$ is fair with respect to the profile $P''$, i.e. $a'$ is achievable for $h$ under the preference profile $P''$; 2. the apartment $a'$ is the best acceptable achievable apartment of $h$ under $P''$, and since $\mu^{NDAI}[P'']$ is Pareto undominated by other fair assignments, we conclude that $\mu^{NDAI}[P'](h) = \mu^{NDAI}[P''](h)$. □

The following Lemma establishes that households are not worse off when a household successfully misrepresents her true preference list.

We are ready to prove that truth-telling is a dominant strategy for all households. We assume, on the contrary, the existence of household $h^*$ and a successful misrepresentation $P'_{h^*}$ of $P_{h^*}$. That is to say $a' P_{h^*} a$, where $\varphi_A^{NDAI}[P](h^*) = a$ and $\varphi_A^{NDAI}[P'_{h^*}, P_{-h^*}] = a'$.

By Lemma Appendix C.6, we consider that $P'_{h^*}$ is the preference list where $a'$ is the most preferred apartment. Our objective is to show that $P'_{h^*}$ is not a successful manipulation. To do that, we follow the proof of Roth [48] about the strategy-proofness of the DA. We say that household $h$ **makes a match** at step $t$ of the NDA algorithm, if $h$ demands $\varphi_A^{NDAI}(h)$ at step $t$. This proof analyzes the possible steps where $h^*$ makes a match.

First, if a household makes a match in the last step of the NDA algorithm, under the true preferences, then no manipulation is a successful misrepresentation of her true preference list.

**Claim Appendix C.3.** Let $x^*$ be the last iteration of the NDAI. Suppose that $h^*$ makes a match at $t^*$, with $1 \leq t^* \leq T$, then $\varphi_A^{NDAI}[P'](h) = \varphi_A^{NDAI}[P](h)$ for all $h$ that makes a match at $T$. Moreover, if $h^*$ makes a match at $T$, there is no profitable deviation $P'_{h*}$ of her true preference list $P_{h^*}$.

*Proof.* First, we present the argument for $h$. Let $T$ be the last step of the $NDA[P]$ and consider that the household $h$ makes a match at step $T$, say $a = \varphi_A^{NDAI}[P](h)$. Since $\mu^{NDAI}[P]$ is non-wasteful, all apartments are matched, at $T-1$ either

1. $a$ is unmatched, or
2. $a$ is matched to a household $h_1$ who is unmatched at $\mu^{NDAI}[P]$.

**Case 1**. Since apartment $a$ was unmatched at $T-1$, all matched households prefer their match at $\mu^{NDAI}[P]$ to $a$. By Lemma Appendix C.2, this implies that none of them gets $a$ at $\mu^{NDAI}[P'_{h^*}, P_{-h^*}]$. By Lemma Appendix C.3 all unmatched households are still unmatched. So, by non-wastefulness, $h$ gets $a$ and does not strictly improve her match under the profile $(P'_{h^*}, P_{-h^*})$.

**Case 2**. Let $h_1$ be the household matched with $a$ at $T-1$, who is unmatched at $T$. For all matched $h_2 \neq h_1$, if any, who prefer $a$ to $\varphi_A^{NDAI}[P](h_2)$, we have that $(a, h_1) \succsim_{\tau(h_1)} (a, h_2)$, by fairness of $\mu^{NDAI}[P]$, or $\tau(h_1)\pi_a\tau(h_2)$. Thus, if $h$ strictly improves under $(P'_{h^*}, P_{-h^*})$, then $h_1$, or an unmatched household, gets $a$ or an apartment preferred to $a$ at $\mu^{NDAI}[P'_{h^*}, P_{-h^*}]$ because the assignment is fair; in contradiction to Lemma Appendix C.3.

In both cases we conclude that $\varphi_A^{NDAI}[P'_{h^*}, P_{h^*}](h) = \varphi_A[P](h)$.

Second, the argument is the same for $h^*$ that makes a match at $T$ since other households $h$ do not improve her allocation under $(P'_{h^*}, P_{-h^*})$ (Lemma A.2). □

Now, we consider that $h^*$ makes a match at some step $t$ of the $NDA[P]$ procedure, with $1 \leq t < T$. In any step where household $h^*$ makes a match, Claim Appendix C.2 implies that $\varphi_A^{NDAI}[P'](h^*) = \varphi_A^{NDAI}[P](h^*)$. That is to say, there is no successful misrepresentation of $P_{h^*}$, and the NDAI is strategy-proof.

**Proof of Theorem 5.2:**

**Distributional Constraints**. See the proof of Theorem 5.1.

**Non-wastefulness.** Follows from the fact that each institution fills its quota.

**There is no justified envy over households of the same type.** Let $x^*$ be the last iteration of the NDAI. Suppose, on the contrary, the existence of a pair $(h, i)$ that has justified envy over a pair $(h', i)$, where $i \in \tau(h) \cap \tau(h')$. Then, there exists an apartment $a$ such that $\varphi_A^{NDA}(h') = a$, $a \in \theta(i)$, and

**i.** $aP_h\varphi_A^{NDA}(h)$,

**ii.** $(a, h) \in Ch_i(\mu^{NDA}(i) \cup \{(a, h)\}, q_i)$.

The proof is analogous to the proof of Case 1 in Theorem 5.1.

Note that Case 2 can not be extended when households are attached to multiple institutions because it a household $h$ can form a blocking pair with another institution $i' \neq i$, which is what happens in example 5.3.

**Pareto Undominated for fair over the same type assignments**. We proceed by induction to show that no household is rejected by an achievable apartment during the NDA algorithm at iteration $x^*$ where there are not interrupters.

*Hypothesis of induction.* At step $t$ of the NDA algorithm, we assume that no household has been rejected by an achievable apartment. In other words, if a household is rejected by some apartment, this apartment is not achievable for her.

*Induction step.* Consider that some household $h^*$ is rejected at step $t+1$ from an apartment $a$. We assume, on the contrary, that $a$ is achievable for household $h^*$. Thus, there exists a fair assignment $\mu = (\theta^\mu, \varphi^\mu)$ such that $\varphi^\mu(h^*) = (a, i^*)$. So, the pair $(a, h^*)$ is acceptable for the institution $i^*$.

Now, let $h$ be the household assigned to the apartment $a$ at the end of step $t+1$, this means that $\varphi^{t+1}(h) = (a, i)$ where $i = \tau(h)$. We analyze the following cases.

**Case 1.** $i \in \tau(h^*)$. Since $\varphi^{t+1}(h) = (a, i)$, the apartment $a$ belongs to $\theta^{t+1}(i)$. That is to say

$$\{(a, h)\} \succ^i \{(a, h^*)\} \tag{C.9}$$

because $(a, h^*) \notin \mu^{t+1}(i)$. Since priorities are responsive, we have that $(a, h) \in Ch_i(\mu(i) \cup \{(a, h)\}, q_i)$. Note that $h$ prefers $a$ to all the apartments that have not rejected her, then the induction hypothesis ensures that household $h$ prefers $a$ to any other achievable apartment for her $aP_h\varphi^\mu_A(h)$. Moreover, $(a, h^*) \in \mu(i)$. That is to say, the pair $(h, i)$ has justified envy over the pair $(h^*, i)$ at the apartment $a$ in the assignment $\mu$, which contradicts the fact that $\mu$ is a fair assignment.

**Case 2.** $i \notin \tau(h^*)$. We know that $\varphi^\mu(h^*) = (a, i^*)$, i.e. the pair $(a, h^*)$ is acceptable for the institution $i^*$, so $i^* \in I_a^{t+1}$. Since there are not interrupters at iteration $x^*$, the offer of $i^*$ is rejected only if $i$ fulfills its quota. So, there exist $(a_{\hat{x}}, h_{\hat{x}}) \in \mu^{t+1}(i^*)$ such that $\{(a_{\hat{x}}, h_{\hat{x}})\} \succ^{i^*} \{(a, h^*)\}$ where $\hat{x} = 1, 2, \ldots, q_{i^*}$. We know that $(a, h^*) \in \mu(i^*)$, i.e., some pair $(a_{\hat{x}}, h_{\hat{x}}) \notin \mu(i^*)$. By hypothesis of induction, this implies that $h_{\hat{x}}$ prefers $a_{\hat{x}}$ to any other achievable apartment for her, i.e.

$$a_{\hat{x}} P_{h_{\hat{x}}} \varphi^\mu_A(h_{\hat{x}}). \tag{C.10}$$

Since $(a_{\hat{x}}, h_{\hat{x}}) \succ^{i^*} \varnothing$, by expression (C.10), we conclude that the pair $(h_{\hat{x}}, i^*)$ has justified envy over the pair $(h^*, i^*)$ in the assignment $\mu$, which contradicts the fact that $\mu$ is fair.

In any case, a contradiction arises when we assume that household $h^*$ is rejected by some achievable apartment $a$. So, no household is rejected by an achievable apartment. Therefore, $\mu^{NDA}$ is Pareto undominated by fair assignments.

**Strategy-proofness**. The prove is analogous to Theorem 5.1. In the following Lemma we generalize the Case 1 of strategy-proofness' proof in Theorem 5.1 when households are attached to multiple institutions.

**Claim Appendix C.4.** Consider a market through institutions with distributional constraints where the over-demand condition holds. Consider a pair $(a, h)$ such that $aP_h\varphi_A^{NDAI}[P](h)$. There is no misrepresentation $P'_h$ of $P_h$ such that $\varphi_A^{NDAI}[P'_h, P_{-h}](h) = a$.

*Proof.* Let $P'_h$ be a misrepresentation of $P_h$ where apartment $a$ is acceptable. We analyze the following cases.

**Case 1**. All institutions in $i \in \tau(h)$ are interrupters for the apartment $a$. As a consequence, all pairs $(a, h)$ are deleted from the priority $\succ^{i,x_i}$, i.e., pairs $(a, h)$ are not acceptable at priority $\succ^{i,x'}$ for all $x' > x_i$. So, household $h$ is not assigned to apartment $a$ in stages $x'.1$, with $x' > x_i$. Therefore, $P'_h$ is not a successful misrepresentation of $P_h$.

**Case 2**. There exists an institution $i \in \tau(h)$ such that $i$ is not an interrupter for apartment $a$. We proceed as in Theorem 5.1 to prove that no misrepresentation of $P_h$ is successful. □

Therefore, Claim Appendix C.4 implies that the NDAI algorithm is strategy-proof. □

## Appendix D. (Not to be included) Comparison with Kamada and Kojima [49]

In this appendix, we show how the Nested Deferred Acceptance algorithm can be applied to the markets with distributional constraints and regional preferences introduced by Kamada and Kojima [49]. This model is inspired by the assignment of doctors to hospitals in Japan. Instead of considering a simple many-to-one matching between hospitals and residents, Kamada and Kojima [6] note that there exists some flexibility in the way hospitals fill their quotas of positions, and introduce "regional preferences" over the pairs of hospitals and doctors. Markets with distributional constraints and regional preferences share some common features with matching markets through institutions. Since doctors are interested in hospitals, hospitals care about doctors and regions care about the number of doctors that each hospital can accept, we note that doctors, regions and hospitals play a similar role as households, institutions and apartments in a market with institutions.

There are two main differences between Kamada and Kojima [49] and our model of matching through institutions. First, hospitals have priorities over doctors in Kamada and Kojima [49] whereas objects have priorities over institutions in our model. Second, and more importantly, regional preferences in Kamada and Kojima [49] are defined over the distribution of doctors over hospitals as measured by a vector of capacities whereas we suppose that institutions care about the precise assignment of households to objects. Notwithstanding those differences, we show that the NDAI algorithm can usefully be applied to produce stable assignments in markets with distributional constraints and regional preferences.

We now define the markets with distributional constraints and regional preferences of Kamada and Kojima [49]. A market with distributional constraints and regional preferences $\tilde{E} = (D, H, Q, R, \tau, P, \succ, \succsim, \tilde{Q})$ is defined by:

1. $D = \{d_1, d_2, \ldots, d_D\}$ is a finite set of doctors, a generic doctor is denoted by $d$;
2. $H = \{h_1, h_2, \ldots, h_H\}$ is a finite set of hospitals, a generic hospital is denoted by $h$;
3. $Q = (q_{h_1}, q_{h_2}, \ldots, q_{h_H})$ is a vectors of quotas, where $q_h$ is the quota of the hospital $h$, a generic quota is $q$;
4. $R = \{1, 2, \ldots, R\}$ is a finite set of regions, a generic region is $r$;

5. $\tau : H \to R$ is the region function, i.e. if a hospital $h$ belongs to the region $r$, we write that $\tau(h) = r$. Let $H_r$ be the set of hospitals in region $r$, note that $H_r \cap H_{r'} = \varnothing$ for region $r' \neq r$;

6. $P = (P_{d_1}, P_{d_2}, \ldots, P_{d_D})$ is the vector of doctors' preferences, $P_d$ is the strict preference of household $h \in H$ over $H \cup \varnothing$; $hP_dh'$ means that doctor $d$ prefers $h$ to $h'$, a hospital $h$ is acceptable for doctor $d$ if $hP_h\varnothing$.

7. $\succ = (\succ_{h_1}, \succ_{h_2}, \ldots, \succ_{h_H})$ is the profile of hospitals priorities over the set of doctors $D$. We assume that for each $h \in H$ the preference $\succ_h$ is responsive on $2^D$, i.e. for any $d, d' \in D$ and $S \in 2^D$ we have that

   **i.** $S \cup \{d\} \succ_h S \cup \{d'\}$ if and only if $d \succ_h d'$, and

   **ii.** $S \cup \{d\} \succ_h S$ if and only if $d \succ_h \varnothing$;

8. $\succsim_r$ is the regional preference of $r$ over the set of vectors $W_r = \{w = (w_h)_{h \in H} | w_h \in \mathbb{Z}_+\}$, where $w_h$ specifies the number of doctors allocated to each hospital $h$ in region $r$;

9. There exists a vector of regional caps $\tilde{Q} = (q_r)_{r \in R}$, where $q_r$ is a non-negative integer for each region $r$.

Kamada and Kojima [49] introduce quasi-choice rules which pick the preferred capacity vector given the regional cap. Given $\succsim_r$, a function $\tilde{Ch}_r : W_r \times \mathbb{Z}_+ \to W_r$ is an **associated quasi choice rule** if $\tilde{Ch}_r(W_r, q_r) \in argmax_{\succeq_r}\{w \in W_r ||w| \leq q_r\}$ for any non-negative $w = (w_h)_{h \in H_r}$. They also require that the quasi choice rule $\tilde{Ch}_r$ be **consistent**, that is, $\tilde{Ch}_r(\omega) \leq \omega' \leq \omega \Rightarrow \tilde{Ch}_r(\omega') = \tilde{Ch}_r(\omega)$. In other words, if $\tilde{Ch}_r$ is still available when the capacity vector reduces to $\omega' \leq \omega$, then the associated quasi-choice rule chooses $\tilde{Ch}_r(\omega')$. They also assume that the regional preferences $\succeq_r$ satisfy the following regularity conditions:

(1) $\omega' \succ_r \omega$ if $\omega_h > q_h \geq \omega'_h$ for some $h \in H_r$ and $\omega'_{h'} = \omega_{h'}$ for all $h' \neq h$. In words, no hospital wants more doctor than its real capacity. This implies that $[\tilde{Ch}_r(w)]_h \leq q_h$ for each $h \in H_r$.

(2) $\omega' \succ_r \omega$ if $\sum_{h \in H_r} \omega_h > q_r \geq \sum_{h \in H_r} \omega'_h$. So, each region prefers the total number of doctors in the region to be at most its regional cap.

(3) If $\omega' \lneq \omega \leq q_{H_r} := (q_h)_{h \in H_r}$ and $\sum_{h \in H_r} \omega_h \leq q_r$, then $\omega \succ_r \omega'$. In other words, each region prefers to fill as many positions of hospitals in the region while the regional cap would not be violated.

Regional preferences $\succsim_r$ are said to be **substitutable** if there exists an associated quasi choice rule $\tilde{Ch}_r$ that satisfies $w \leq w' \Rightarrow \tilde{Ch}_r(w) \geq \tilde{Ch}_r(w') \wedge w$.

Next, Kamada and Kojima [49] define stable matchings in markets with distributional constraints and regional preferences:

A **matching** $\mu$ is a function that satisfies

(i) $\mu(d) \in H \cup \{\varnothing\}$ for all $d \in D$,

(ii) $\mu(h) \subseteq D$ for all $h \in H$ and

(iii) for any $d \in D$ and $h \in H$, $\mu(d) = h$ if and only if $d \in \mu(h)$.

A matching is **feasible** if $\mu(r) \leq q_r$ for all $r \in R$, where $\mu_r = \bigcup_{h \in H_r} \mu(h)$.

A matching $\mu$ is **stable** if it is feasible, individually rational, and if $(d, h)$ is a blocking pair, then

(i) $|\mu(h)| = q_{r_h}$,

(ii) $d' \succ_h d$ for all doctors $d' \in \mu(h)$, and

(iii) either $\mu(d) \notin H_{r(h)}$ or $w \succsim_{r(h)} w'$,

where $w_{h'} = |\mu(h')|$ for all $h' \in H_{r(h)}$ and $w'_h = w_h + 1$, $w'_{\mu(d)} = w_{\mu(d)} - 1$ and $w'_{h'} = w_{h'}$ for all other $h' \in H_{r(h)}$.

We adapt the NDA algorithm to the model of Kamada and Kojima [49]. The algorithm differs from the NDA algorithm of our baseline model in two respects: (i) unmatched doctors are selected sequentially rather than simultaneously to make offers and (ii) we replace the choice function of regions by the quasi-choice function of regions. Formally:

**Initialization**

Consider a market $(D, H, Q, R, \tau, P, \succ, \succsim, \tilde{Q})$ with distributional constraints and regional preferences. The matching is initialized to be the empty matching, so $\mu^0(h) = \mu^0(d) = \mu^0(r) = \varnothing$, for all $d \in D$, $h \in H$ and $r \in R$.

For all doctors $d \in D$, let $H_d^t := H$, and $t = 1$. For each region $r$, fix a quasi-choice rule $\tilde{Ch}_r$.

**$\mathbf{A}^t$. Eliciting the demand of doctors**

Arbitrarily pick one unassigned doctor $d$, who asks for the most preferred hospital in $H_d^t$, denoted by $D_d^t$ while $r$ is the region of $D_d^t$; moreover matched doctors $d'$ in region $r$ iterate their demand to their match, $D_{d'}^t = \{\mu^{t-1}(d)\}$.

For all hospitals $h \in H$ in region $r$, we define the set of doctors that demand hospital $h$ in region $r$ as follows:

$$D_{h,r}^t = \{d \in D | D_d^t = \{h\}, d \succ_h \varnothing \text{ and } \tau(h) = r\}.$$

The set of pairs $(d, h)$ that can be assigned to region $r$ is defined as

$$M_r^t = \{(d, h) \in D \times H | d \succ_h \varnothing \text{ and } d \in D_{h,r}^t\}.$$

The possible assignments are

$$\mathcal{P}_r^t = \left\{p = \{(d, h)\}_{\tau(h)=r} \mid (d, h) \in M_r^t \text{ and } d \text{ is not matched twice}\right\}.$$

The number of doctors matched to hospital $h$ at $p$ is $w_h(p) = |\{d \in D \mid (d, h) \in p\}|$, thus, the set of capacity vectors is

$$W_r^t = \left\{w = (w_h)_{h \in H} | \exists\, p \in \mathcal{P}_r^t \text{ and } w_h = w_h(p) \text{ for all } h \text{ in region } r\right\}.$$

**B$^t$. Matching the demand of region r and hospitals of the region.**

**B$^t$.1** Regions $r$ demands the vector

$$\omega_r^t = (\omega_h^t)_{\tau(h)=r} = \tilde{Ch}_r(W_r^t, q_r).$$

**B$^t$.2** Each hospital $h$ in region $r$ is tentatively assigned to the preferred subset of $D_{h,r}^t$ with cardinality $w_h^t$. The assignment in other regions remains the same.

**C$^t$. Iteration over $\mathbf{D_d^t}$**

Let $H_d^{t+1} := H_d^t \setminus \{\max_{P_d} H_d^t\}$, $t := t+1$.

If all doctors have been rejected by all the apartments in her preference list or is matched, the tentative assignment becomes the outcome assignment. Otherwise, go to the Phase A$^{t+1}$.

The assignment produced by the previous algorithm is denoted by $\tilde{\mu}^{NDA}$. It depends on a market $\tilde{E}$ and a fixed associated quasi choice rule $\tilde{Ch}$. We now state that the NDA produces a stable and strategy-proof matching in the market with distributional constraints and regional preferences:

**Theorem Appendix D.1.** *Suppose that regional preferences $\succeq_r$ are substitutable for all $r \in R$. Then the matching produced by the nested deferred acceptance algorithm is stable and strategy proof for doctors.*

*Proof.* We adapt the proof of Kamada and Kojima [49] to the Nested Deferred Acceptance algorithm with regional preferences.

First, as they do, we establish the relation between matching markets with regional preferences and matching with contracts. So, let $X = D \times H$ be the set of contracts. Note that, for each doctor $d$, the preference profile $P_h$ induces a preference relation $\tilde{P}_h$ over $(\{d\} \times H) \cup \{\varnothing\}$ in the following way $(d, h')\tilde{P}_d(d, h)$ if and only if $h' P_d h$. Moreover, we say that $(d, h)\tilde{P}_d \varnothing$ if hospital $H$ is unacceptable under $P_d$.

Now, for each region $r \in R$, we define preferences $\succeq_r$ and its associated choice rule $\overline{Ch}_r$ over all subsets of $D \times H_r$. For any $X' \subset D \times H_r$, let $\omega(X') := (w_h(X'))_{h \in H_r}$ be the vector such that $w_h(X') = |\{(d, h) \in X' | d \succ_h \varnothing\}|$. For each $X'$, the chosen set of contracts $\overline{Ch}_r(X')$ is defined by

$$\overline{Ch}_r(X') = \bigcup_{h \in H_r} \left\{(d,h) \in X' \middle| |\{d' \in D | (d', h) \in X', d' \succeq_h d\}| \leq (\tilde{Ch}_r(\omega(X')))_h \right\}. \tag{D.1}$$

That is, each hospital $h \in H_r$ chooses its $(\tilde{Ch}_r(\omega(X')))_h$ most preferred contracts available in $X'$. The domain of the choice rule $\overline{Ch}_r$ can be extended to all subsets of $X$ by

$$\overline{Ch}_r(X') = \overline{Ch}_r(\{(d,h) \in X' | h \in H_r\})$$

for any $X' \subseteq X$.

**Definition 2.** Hatfield and Milgrom [3]. Choice rule $\overline{Ch}_r(\cdot)$ satisfies the **substitutes condition** if there does not exist contracts $x, x' \in X$ and a set of contracts $X' \subseteq X$ such that $x' \notin \overline{Ch}_r(X' \cup \{x'\})$ and $x' \in \overline{Ch}_r(X' \cup \{x, x'\})$.

**Definition 3.** Hatfield and Milgrom [3]. Choice rule $\overline{Chr}(\cdot)$ satisfies the **law of aggregate demand** if for all $x' \subseteq X'' \subseteq X$, $|\overline{Ch}_r(X')| \leq |\overline{Ch}_r(X'')|$.

**Proposition Appendix D.1.** *Suppose that $\succeq_r$ satisfies the substitutes condition. Then the choice rule $\overline{Ch}_r(\cdot)$ defined above satisfies the substitutes condition and the law of aggregate demand.*

*Proof.* The Nested Deferred Acceptance algorithm is not related with the choice rule $\overline{Ch}_r$. So, this proposition is taken from Proposition 1 of Kamada and Kojima [49]. □

A subset $X'$ of $X = D \times H$ is said to be **individually rational** if (1) for any $d \in D$, $|\{(d,h) \in X' | h \in H\}| \leq 1$, and if $(d,h) \in X'$ then $hP_d\varnothing$, and (2) for any $r \in R$, $\overline{Ch}_r(X') = X' \cap (D \times H_r)$.

**Definition 4.** A set of contracts $X' \subseteq X$ is a **stable allocation** if

(1) it is individually rational, and

(2) there exists no region $r \in R$, hospital $h \in H_r$, and a doctor $d \in D$ such that $(d,h)\tilde{P}_d x$ and $(d,h) \in \overline{Ch}_r(X' \cup \{(d,h)\})$, where $x$ is the contract that $d$ receives at $X'$ if any and $\varnothing$ otherwise.

When condition (2) is violated by some $(d,h)$, we say that $(d,h)$ is a **block** of $X'$. A doctor-optimal stable allocation in the matching model with contracts is a stable allocation that every doctor weakly prefers to every other stable allocation [3].

Given any individually rational set of contracts $X'$, define a corresponding matching $\mu(X')$ in the original model by setting $\mu(d)(X') = h$ if and only if $(d,h) \in X'$; and $\mu(d)(X') = \varnothing$ if and only if no contract associated with $d$ is in $X'$. Since each doctor regards any set of contracts with cardinality of at least two as unacceptable, each doctor receives at most one contract at $X'$ and hence $\mu(X')$ is well defined for any individually rational $X'$.

**Proposition Appendix D.2.** *If $X'$ is a stable allocation in the associated model with contracts, then the corresponding matching $\mu(X')$ is a stable matching in the original model.*

*Proof.* See Proposition 2 of Kamada and Kojima [49]. □

**Remark 1.** It is is important to recall the connection between the Nested Deferred Acceptance and the Cumulative Offer Process of Hatfield and Milgrom [3]. That is to say, if doctor $d$ asks for her most preferred hospital $h$ at some step in the NDA, then contract $(d,h)$ is proposed at the same type of the cumulative offer process. Also, the set of doctors accepted by a hospital at some step of the NDA is equivalent to the set of contracts held at the corresponding step of the cumulative offer process. Thus, if $X'$ is the output of the cumulative offer process, then $\mu(X')$ is the matching generated by the NDA.

Now, we are ready to continue with the proof of Theorem 6.1. By Proposition 1, the choice function of each region satisfies the substitutes condition and the law of aggregate demand in the associate model of matching with contracts. By Hatfield and Milgrom [3], Hatfield and Kojima [14], and Hatfield and Kominers [22], the cumulative offer process with choice functions satisfying these conditions produces a stable allocation and is strategy-proof. The former fact, together with Remark 1 and Proposition Appendix D.2, implies that the outcome of the Nested Deferred Acceptance Algorithm is a stable matching in the original model. By Remark 1, we conclude that the NDA mechanism is strategy-proof for doctors. □

In order to find an assignment between hospitals and doctors that respect the distributional constraints and regional caps, Kamada and Kojima introduced the Generalized Flexible Deferred Acceptance (GFDA) algorithm.

For each region $r$ fix a quasi-choice rule $\tilde{Ch}_r$. The GFDA algorithm proceed as follows

1. Begin with an empty matching, i.e. $\mu_d = \varnothing$ for all $d \in D$.
2. Choose a doctor $d$ arbitrarily who is currently not tentatively matched to any hospital and who has not applied to all acceptable hospitals yet. If such a doctor does not exist, then terminate the algorithm.
3. Let $d$ apply to the most preferred hospital $\overline{h}$ at $H_d$ among the hospitals that have not rejected d so far. If $d$ is unacceptable to $\overline{h}$, then reject this doctor and go back to step 2. Otherwise, let $r$ be the region such that $\overline{h} \in H_r$ and define vector $\omega = (\omega_h)_h \in H_r$ by

   **(a)** $\omega_{\overline{h}}$ is the number of doctors currently held at $\overline{h}$ plus one, and

   **(b)** $w_h$ is the number of doctors currently held at $h$ if $h \neq \overline{h}$,

4. Each hospital $h \in H_r$ considers the new applicant $d$ (if $h = \overline{h}$) and doctors who are temporarily held from the previous step together. It holds its $(\tilde{Ch}_r(w))_h$ most preferred applicants among them temporarily and rejects the rest (so doctors held at this step may be rejected in later steps). Go back to step 2.