Lecture V: Duality and Welfare Evaluations

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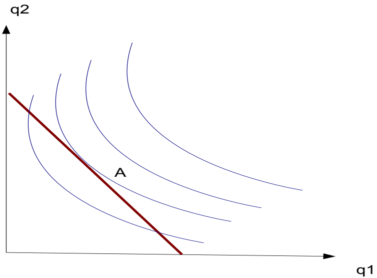
Utility maximization



Utility maximization

- Let the prices p be fixed, and the income x be fixed.
- The budget constraint is then fixed: $p_1 q_1 + p_2 q_2 = x$
- Consumer select the optimal choice (point A) as a solution to:

$$\max u(q_1, q_2) \text{ s to } p_1 q_1 + p_2 q_2 = x$$

(Marshallian) demand functions

- The solution is the (Marshallian) demand function

$$g_i(p_1, p_2, x)$$

- This demand function depends on prices and income.
- It is homogeneous of degree zero in prices and income (same demand function if p_1,p_2 and x are multiplies by the same proportionality factor α)
- It is increasing in x if good i is normal
- It is decreasing in p_i if good i is normal (law of demand)

Indirect utility

- Define indirect utility (as a function of p and x) as the maximal utility,

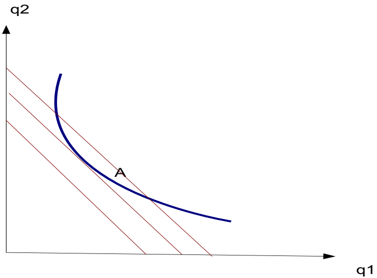
$$V(x,p) = u(g_1(p,x), g_2(p,x)) = \max_{q | q_1 + p_2 q_2 = x} u(q_1, q_2),$$

- Indirect utility is homogeneous of degree zero in prices and income
- Indirect utility is increasing in x (as the budget set increases, the utility level must go up)
- Indirect utility is decreasing in p_1 and p_2 (as the budget set decreases, the utility level must go down)

Expenditure minimization

- Consider the following problem: Let prices p_1 and p_2 be fixed.
- What is the lowest level of income needed to reach the utility level U?
- This problem is called the *expenditure minimization problem*

utility maximization



(Hicksian) demand functions

- The solution to the expenditure minimization problem is called the *Hicksian or compensated demand function* $h(p_1, p_2, u)$
- The Hicksian demand function is defined for fixed prices and *utility levels*
- The Hicksian demand functions are homogeneus of degree zero in prices

Expenditure functions

- The expenditure function is:

$$C(p,u) = p_1 h(p,u) + p_2 h(p,u) = \min_{q|u(q)=u} p_1 q_1 + p_2 q_2$$

- The expenditure function is homogeneous of degree one in prices (multiplying both prices by α multiplies expenditure by the same factor α)
- The expenditure function is increasing in u (in order to reach a higher indifference curve, at fixed prices, income must go up)

Duality

- There is a relation between utility maximization and expenditure minimization:

$$
\begin{aligned}
V(p, C(p, u)) &= u, \\
C(p, V(p, x)) &= x.
\end{aligned}
$$

- There is a relation between Marshallian and Hicksian demand functions:

$$
\begin{aligned}
g(p, C(p, u)) &= h(p, u), \\
h(p, V(p, x)) &= g(p, x)
\end{aligned}
$$

Shephard's Lemma

- Shephard's Lemma establishes a relation between the exenditure function and the Hicksian demand function:
$$\frac{\partial C(p,u)}{\partial p_i} = h_i(p,u).$$
- **Proof:** The Lagrangian of the expenditure minimization problem is:
$$\mathcal{L} = p_1 q_1 + p_2 q_2 + \lambda(u - u(q_1, q_2))$$
Differentiating with respect to q_i,
$$p_i = \lambda \frac{\partial u}{\partial q_i}$$
- Compute
$$\frac{\partial C(p,u)}{\partial p_i} = h_i(p,u) + p_1 \frac{\partial q_1}{\partial p_1} + p_2 \frac{\partial q_2}{\partial p_1}.$$

Roy's identity

- Roy's identity establishes a relation between the indirect utility function and the Marshallian demand function:

$$g_i(p,x) = -\frac{\frac{\partial V}{\partial p_i}}{\frac{\partial V}{\partial x}}.$$

- **Proof:** We will use Shepards' Lemma. We use the duality result:

$$V(p, C(p,u)) = u.$$

Deriving the equality with respect to p_i:

$$\frac{\partial V}{\partial p_i} + \frac{\partial V}{\partial x}\frac{\partial C}{\partial p_i} = 0.$$

- Using Shepards's Lemma, $\frac{\partial C}{\partial p_i} = h_i(p,u) = g_i(p,x)$ yielding the result.

Welfare evaluations

- Using the theory of preferences, we can ask the *normative question*: How do individual consumers benefit or suffer from economic changes?
- We analyze the effect of a change from a situation (p^0, x^0) to a situation (p^1, x^1).
- The obvious answer is to use the indirect utility function:
- the consumer is better (or worse) off if $V(x^1, p^1) \geq V(x^0, p^0)$.
- This gives us an easy *ordinal* way to say whether the consumer is better off or not.
- We also want to *measure* by how much the consumer is better off.

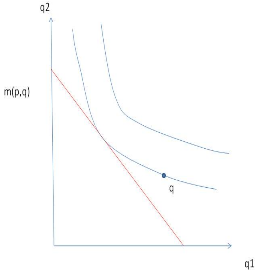
Direct Compensation Function

- Our ultimate goal is to ”measure” consumer well-being in a way which allows us to easily compute effects of changes in the environment.
- Suppose that a consumer currently consumes a bundle q and receives utility $u(q)$.
- How much money should we give her so that she is at least as well off, when the prices are p?
- In order to answer this auestion, we define the *direct compensation function*

$$m(p, q) \equiv C(p, u(q)).$$

- *It is also called a money metric utility function*.

Direct compensation function



Properties of direct compensation functions

- This is a function which depends on q (like a utility function), but also on p (the price vector)
- The function gives a ”money unit”, so it is defined as a ”money metric utility function”
- For fixed q, it inherits the properties of the expenditure function: monotonic, homogeneous in p
- for fixed p, it is in fact a utility function

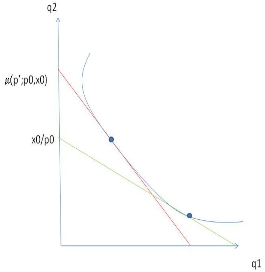
Indirect compensation functions

- We can define an indirect compensation function, using the indirect utility function:
- Suppose that initially the consumer faces prices p and an income x, and has an indirect utility $V(p, x)$.
- If prices are p', then we compute the *indirect compensation function*

$$\mu(p'; p, x) = C(p', V(x, p)).$$

- This is again a money metric function: it behaves like an expenditure function with respect to p' and like and indirect utility function with respect to p and x.

Indirect compensation function



Equivalent and compensating variation

- Suppose that the price and income change from (p^0, x^0) to (p^1, x^1).
- The indirect compensation function gives us an easy way to measure the effect of a welfare change, by measuring:

$$\mu(p'; p^1, x^1) - \mu(p'; p^0, x^0)$$

for *some* base price p'.

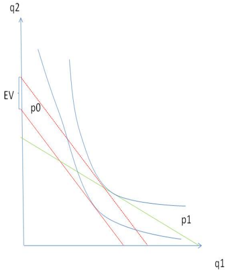
- Following Hicks (1939), there are two obvious choices for the base price: the initial price p^0 and the final price p^1,
- *The equivalent variation* is defined using the initial price,

$$EV = \mu(p^0; p^1, x^1) - \mu(p^0; p^0, x^0) = \mu(p^0; p^1, x^1) - x^0$$

- *The compensating variation* is defined using the final price:

$$CV = \mu(p^1; p^1, x^1) - \mu(p^1; p^0, x^0) = x^1 - \mu(p^1; p^0, x^0)$$

Equivalent variation

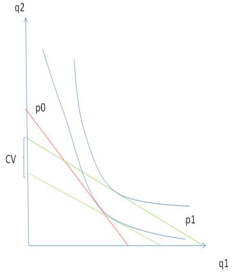


Equivalent variation

- The equivalent variation is a notion ”ex ante”: how much would the agent accept to give (or ask to receive) today so that the proposed change does not occur?
- We can as well compute:

$$V(p^0, x^0 + EV) = V(p^1, x^1)$$

Compensating variation



Compensating variation

- The compensating variation is a notion ”ex post”: how much does the agent accept to give (or asks) in order to be compensated tomorrow after the proposed change has occurred?
- We can as well compute:

$$V(p^1, x^1 - CV) = V(p^0, x^0)$$

Welfare evaluation of a price change

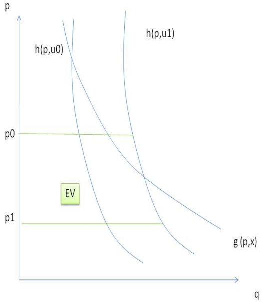
- EV and CV are mostly used to analyze the effect of a price change from p^0 to p^1, while the income level x remains fixed.
- We now consider this special case, and let $u^0 = V(p^0, x)$ and $u^1 = V(p^1, x)$ denote the utility levels achieved before and after the price change at the income level x.
- We have: $EV = C(p^0, u^1) - x$ and $CV = x - C(p^1, u^0)$.

Hicksian Demands, Equivalent and Compensating Variations

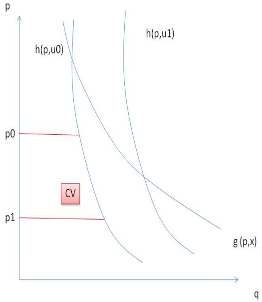
- Recalling that $h_1(p,u) = \frac{\partial C(p,u)}{\partial p_1}$,

$$
\begin{aligned}
EV(p^0, p^1, x) &= C(p^0, u^1) - x, \\
&= C(p^0, u^1) - C(p^1, u^1), \\
&= \int_{p^1}^{p^0} h_1(p, u^1) dp, \\
CV &= \int_{p^1}^{p^0} h_1(p, u^0) dp
\end{aligned}
$$

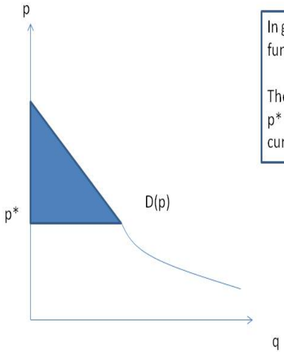
Equivalent Variation



Compensating Variation



Consumer surplus



In general, the demand function D(p) is decreasing.

The **consumer surplus** at price p* is the area below the demand curve D(p) and above the price p*

Consumer surplus

- If the consumer faces a price p^*, she buys all units at that price.
- Some units are valued higher than p^*. These are all the units on the demand curve above p^*.
- The consumer "gains" on all those units. The sum of these gains is the *consumer surplus*: the area above price p^* and below the demand curve $D(p)$.

$$CS = \int_{p}^{\infty} D(p)dp.$$

Quasilinear utility

- Utility is quailinear if $u(q_1, q_2) = q_1 + u(q_2)$.
- The good q_1 is then called the 'numeraire'
- The consumer chooses first the quantity q_2 such that $u'(q_2) = \frac{p_2}{p_1}$ and adjusts his numeraire holdings according to the income level, $q_1 = \frac{x - p_2 q_2}{p_1}$.
- *In particular, the Marshallian demand of all goods (but the numeraire) is independent of income, so that the Hicksian demand is independent of the utility level*

Equivalence between EV, CV and consumer surplus for quasilinear utilities

- Fix the price of the numeraire to 1 and consider a change from p^0 to p^1 for the prices of other goods
- If utility is quasi linear, $h(p,u)$ only depends on p, and $h(p,u^0) = h(p,u^1) = h(p) = g(p)$.
- We thus have:

$$EV = CV = \int_{p^1}^{p^0} g(p)dp$$

- The equivalent variation and compensating variations coincide, and are equal to the consumer surplus.

Summary of Lecture V

- Expenditure minimization is the problem faced by consumers who want to minimize expenditure to reach a fixed utility target.
- Hicksian (or compensated) demand arises when consumers minimize expenditure in order to reach a utility target.
- indirect utility functions measure the maximal utility as a function of prices and income ; expenditure functions measure the minimal expenditure as a function of prices and utility.
- Duality establishes a relation between expenditure and indirect utility, and between Marshallian and Hicksian demands.
- Shephard's Lemma establishes a relation between Hicksian demand and expenditure functions.

Summary of Lecture V

- In order to measure welfare, we can use money metric utilities (direct and indirect compensation functions)
- The equivalent variation measures how much a consumer is willing to pay to remain at a status quo
- The compensating variation measures how much a consumer needs to be compensated in order to accept a move.
- When utility is quailinear, the effect of a chneg in prices is the same under equivalent and comepnsating variations, and the welfare difference can be measured using consumer surplus.