# Lecture IV: Mechanism Design II

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# Assumptions

- Types are taken from a continuous distribution $P_i(\cdot)$ over $[\underline{\theta}_i, \overline{\theta}_i]$
- Private values: utilities only depend on own type $u_i(x, t_i, \theta_i)$
- Quasi-linear utilities

$$u_i(x, t_i, \theta_i) = V_i(x, \theta_i) + t_i,$$

- either

$$u_0(x, \theta) = V_0(x, \theta) - \sum_i t_i, \text{ self-interested principal}$$

- or

$$u_0(x, \theta) = \sum_i V_i(x, \theta_i), \text{ benevolent principal}$$

- An allocation $x(\theta)$ is *efficient* if it maximizes $\sum_i V_i(x, \theta_i)$ for all $\theta$

# Budget balance

- A mechanism satisfies *budget balance* if the mechanism designer does not need to spend additional money, i.e.

$$\sum_i t_i(\theta) \leq -C_0(x(\theta)) \text{ for all } \theta.$$

- A mechanism $y = (x, t)$ is *feasible* if it is implementable and individually rational
- It is feasible under budget balance if in addition it satisfies budget balance
- If budget balance is not required, it is always possible to make the mechanism individually rational by allowing for arbitrarily large transfers to the agents..

# Dominant strategy versus Bayesian implementation

- A mechanism is a dominant strategy mechanism if

$$u_i(y(\theta_i\theta_{-i}), \theta_i) \geq u_i(y(\hat{\theta}_i, \theta_{-i}), \theta_i) \forall \theta_i, \hat{\theta}_i, \theta_{-i}.$$

- This definition requires that, for all types of the other agents, announcing the true type is a dominant strategy
- A mechanism is a Bayesian mechanism if

$$E_{\theta_{-i}} u_i(y(\theta_i\theta_{-i}), \theta_i) \geq E_{\theta_{-i}} u_i(y(\hat{\theta}_i, \theta_{-i}), \theta_i) \forall \theta_i, \hat{\theta}_i,$$

- The condition need only be satisfied *on average* for the types $\theta_{-i}$
- Clearly Bayesian incentive compatibility is a weaker notion than dominant strategy incentive compatibility (or strategy-proofness)

# An efficiency result: The Groves-Clarke mechanism

- In the context of public goods, Groves (1973) and Clarke (1971) discover an efficient mechanism as long as budget balance is not required.
- This mechanism is implementable in dominant strategies.
- The idea: to align the surplus of agent $i$ and the total surplus, add to the payoff of agent $i$ a transfer which is equal to the sum of the surplus of all other agents.
- The transfers can thus be expressed as ”externalities payment” that the decision creates on all other players
- However it is clear that these transfers are not necessarily budget balanced!

# Groves mechanisms

- Formally, let $x^*(\theta)$ be an efficient allocation under type profile $\theta$,i.e. $x^*(\theta)$ maximizes $\sum_i V_i(x, \theta_i)$.
- Define transfers

$$t_i(\hat{\theta}) = \sum_{j \neq i} V_j(x^*(\hat{\theta}, \hat{\theta}_j) + \tau_{-i}(\hat{\theta}_{-i}),$$

- where $\tau_{-i}$ is an arbitrary function which only depends on the announcements of the other types.
- We claim that it is a dominant strategy for agent $i$ to announce his true type $\theta_i$:

$$V_i(\hat{\theta}_i, \theta_{-i}) = V_i(x(\hat{\theta}_i, \theta_{-i}, \theta_i) + \sum j \neq i V_j(x(\hat{\theta}_i, \theta_{-i}, \theta_j) + \tau_{-i}(\theta_{-i})$$

- Because $x^*(\theta)$ maximizes $V_i + \sum_{j \neq i} V_j$, player $i$ does not have an incentive to lie..

# The Clarke mechanism

- Consider the binary choice between building a project of a fixed size and not building it
- Agent's utilities are $u_i = \theta x_i + t_i$ where $x \in \{0, 1\}$
- Let $c$ be the cost of the public good. The optimal strategy is

$$x^*(\theta) = \begin{cases} 1 & \text{if } \sum_i \theta_i \geq c \\ 0 & \text{otherwise,} \end{cases}$$

- Transfers are then

$$t_i(\theta) = \begin{cases} \sum_j \theta_j - c & \text{if } \sum_i \theta_i \geq c, \\ 0 & \text{otherwise,} \end{cases}$$

# The Clarke mechanism

- This is called the ”pivot” mechanism (and was proposed by Clarke). Agent $i$'s payment is independent of his type except in the region where $i$'s announcement is pivotal and changes the decision
- The pivot mechanism always generates a budget surplus for the planner.

# The AGV mechanism

- The AGV mechanism was discovered by d'Aspremont Gerard Varet (1977) and Arrow (1979)
- It is a Bayesian version of the Groves mechanism and *allows for budget balance*
- The agent's transfer is

$$t_i(\theta_i) = E_{\theta_{-i}} \sum_{j \neq i} V_j(x^*(\theta_i, \theta_{-i}), \theta_j) + \tau_{-i}(\theta_{-i})$$

- Under this transfer rule when $x^*(\theta)$ is efficient, the Bayesian incentive compatibility condition is satisfied..
- We now construct a function $\tau_{-i}(\theta_{-i})$ such that budget balance is satisfied..

# The AGV mechanism

- We now compute transfers so that they are budget balanced
- Let $B_i(\theta_i) = E_{\theta_{-i}} \sum_{j \neq i} V_j(x^*(\theta_i, \theta_{-i}), \theta_j)$
- $B_i(\theta_i)$ only depends on the announcement of agent $i$
- We share $-B_i(\theta_i)$ among all $I - 1$ agents $j \neq i$.
- Doing that, we compute

$$\tau_i(\theta_{-i}) = -\frac{\sum_{j \neq i} B_j(\theta_j)}{I - 1}.$$

- And by construction the transfers are budget balanced.

# Efficiency vs Inefficiency

- We just saw that one can implement efficient allocations in public good economies without budget balance (and with interim budget balance in the Bayesian setting)
- We now describe an environment where budget balance cannot be achieved: the two-agent trading game (similar to the model of double auctions of Chatterjee and Samuelson)

# The Myerson-Satterthwaite environment

- A two-agent trading game
- Seller has a cost $c$ drawn from the distribution $P_1$ over $[\underline{c}, \overline{c}]$
- Buyer has a value $v$ drawn from the distribution $P_2$ over $[\underline{v}, \overline{v}]$
- Let $x(c, v)$ be the probability of trade and $t(c, v)$ the transfer from buyer to seller
- $t_1 \equiv t, t_1 + t_2 = 0$ by construction
- As opposed to the double-auction setting of Chatterjee and Samuelson, we do not impose a particular bargaining protocol.
- Instead we ask a general question about existence of efficient, incentive compatible, individually rational, budget balanced mechanisms..

# The trading game

- Let $X_1(c) = E_v[x(c,v)]$ and $X_2(v) = E_c[x(c,v)]$ denote the seller's and buyer's expected probabilities of trading
- Let $T_1(c) = E_v[t(c,v)]$ and $T_2(v) = -E_c[t(c,v)]$ denote the expected transfers.
- And let the utilities be

$$\begin{aligned} U_1(c) &\equiv T_1(c) - cX_1(c) \\ U_2(v) &\equiv vX_2(v) + T_2(v) \end{aligned}$$

# Myerson-Satterthwaite

- We know that, for incentive compatibility, $X_1(\cdot)$ and $X_2(\cdot)$ must be monotonic: $X_1$ is nonincreasing and $X_2$ is nondecreasing
- Using the "Mirrlees trick" we compute

$$
\begin{aligned}
U_1(c) &= U_1(\overline{c}) + \int_c^{\overline{c}} X_1(\gamma) d\gamma, \\
U_2(v) &= U_2(\underline{v}) + \int_{\underline{v}}^{v} X_2(\nu) d\nu
\end{aligned}
$$

- Substituting for $U_1$ and $U_2$ and adding up

$$
\begin{aligned}
T_1(c) + T_2(v) &= cX_1(c) - vX_2(v) + U_1(\overline{c}) + U_2(\underline{v}) \\
&+ \int_c^{\overline{c}} X_1(\gamma) d\gamma + \int_{\underline{v}}^{v} X_2(\nu) d\nu
\end{aligned}
$$

# Myerson-Satterthwaite

- By budget balance

$$E_c T_1(c) + E_v T_2(v) = 0.$$

- Now

$$\begin{aligned} 0 &= \int_{\underline{c}}^{\overline{c}} [cX_1(c) + \int_c^{\overline{c}} X_1(\gamma)d\gamma] p_1(c)dc + U_1(\overline{c}) \\ &+ \int_{\underline{v}}^{\overline{v}} [\int_{\underline{v}}^{v} X_2(\nu)d\nu - vX_2(v)] p_2(v)dv + U_2(\underline{v}) \end{aligned}$$

# Myerson-Satterthwaite

- Integrating by parts yields

$$
\begin{aligned}
U - 1(\overline{c}) + U_2(\underline{v}) &= -\int_{\underline{c}}^{\overline{c}} [c + \frac{P_1(c)}{p_1(c)}] X_1(c) p_1(c) dc \\
&+ \int_{\underline{v}}^{\overline{v}} [v - \frac{1 - P_2(v)}{p_2(v)}] X_2(v) p_2(v) dv
\end{aligned}
$$

- Replacing $X_1(c)$ and $X_2(v)$

$$
\begin{aligned}
U_1(\overline{c}) + U_2(\underline{v}) &= \int_{\underline{c}}^{\overline{c}} \int_{\underline{v}}^{\overline{v}} [v - \frac{1 - P_2(v)}{p_2(v)} - c - \frac{P_1(c)}{p_1(c)}] \\
&\quad x(c, v) p_1(c) p_2(v) dc dv
\end{aligned}
$$

# The Myerson Satterthwaite inefficiency result

- By IR $U_1(\overline{c}) \geq 0$ and $U_2(\underline{v}) \geq 0$.
- Efficiency requires that $x(c, v) = 1$ if $v > c$
- But this results in a negative term $[v - \frac{1-P_2(v)}{p_2(v)} - c - \frac{P_1(c)}{p_1(c)}]x(c, v)p_1(c)p_2(v)dcdv$

## Theorem

*Suppose that the seller's cost and buyer's valuation have differentiable strictly positive densities on $[\underline{c}, \overline{c}]$ and $[\underline{v}, \overline{v}]$ that $\underline{c} < \overline{v}$ and $\overline{c} > \underline{v}$. Then there is no efficient trading outcome that satisfies individual rationality, incentive compatibility and budget balance.*

# Efficiency vs optimality

- Recall that the planner can have two objectives: (i) efficiency, maximizing the payoff of all agents or (ii) revenue maximization (e.g. a seller's revenue in an auction)
- In the previous section we asked: When is it the case that efficient mechanisms can be constructed?
- Now we consider the (second) best mechanisms which can be constructed, by explicitly computing the optimal mechanism
- We focus on two environments: auctions and bilateral trade.

# Auctions

- The seller has $\hat{x}$ units of a good to sell
- There are $I$ potential buyers, $i = 1, ..I$
- Each agent has quasi-linear utilities

$$u_1 = V_i(x_i, \theta_i) + y_i,$$

  where $x_i \in [0, \hat{x}]$ is the units consumed by player $i$.
- We assume that single crossing holds: $\frac{\partial^2 V_i}{\partial x_i \partial \theta_i} \geq 0$.
- The seller's parameter $\theta_0$ is common knowledge and the seller gets $t_0 = -\sum_{i=1}^{I} t_i$.

# The seller's revenue maximization problem

- The seller maximizes revenues

$$R = E_\theta[V_0(\hat{x} - \sum_{i=1}^{I} x_i(\theta), \theta_0) - \sum_{i=1}^{I} t_i(\theta).$$

- subject to

$$\begin{aligned}(IC) E_{\theta_{-i}}[V_i(x_i(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i})] \quad \geq \quad & E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i) \\ & + t_i(\hat{\theta}_i, \theta_{-i})]\end{aligned}$$

$$(IR) E_{\theta_{-i}}[V_i(x_i(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i})] \geq 0$$

- $x_i(\theta) \geq 0, \sum_{i=1}^{I} x_i \leq \hat{x}$

# Optimal auctions

- Let $U_i(\theta_i) = E_{\theta_{-i}}[V_i(x_i(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i})]$ denote the indirect utility of bidder $i$
- Eliminating transfers in the expression of the seller's revenue

$$R = E_\theta[V_0(\hat{x} - \sum_{i=1}^{I} x_i(\theta), \theta_0) + \sum_{i=1}^{I} V_i(x_i(\theta), \theta_i)] - \sum_{i=1}^{I} E_{\theta_i} U_i(\theta_i).$$

- Using the envelope theorem,

$$\frac{dU_i}{d\theta_i} = E_{\theta-i}[\frac{\partial V_i}{\partial \theta_i}(x_i(\theta), \theta_i)].$$

# Optimal auctions

- So

$$U_i(\theta_i) = U_i(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} E_{\theta-i}[\frac{\partial V_i}{\partial \theta_i}(x_i(\tilde{\theta}_i, \theta_{-i}), \tilde{\theta}_i)]d\tilde{\theta}_i$$

- At the optimum $U_i(\underline{\theta}) = 0$. Integrating by parts,

$$\begin{aligned} R \equiv \; & E_\theta[V_0(\hat{x} - \sum_{i=1}^{I} x_i(\theta), \theta_0) \\ & + \sum_{i=1}^{I}[V_i(x(\theta), \theta_i) - \frac{1 - P_i(\theta_i)}{p_i(\theta_i)} \frac{\partial V_i}{\partial \theta_i}(x_i(\theta), \theta_i) \end{aligned}$$

# Optimal auctions

- Suppose that $V_i(x_i, \theta_i) = \theta_i x_i$
- Suppose that $\hat{x} = 1$
- Then the revenue writes

$$R = E_\theta[\sum_{i=1}^{I}(\theta_i - \frac{1 - P_i(\theta_i)}{p_i(\theta_i)}]x_i(\theta) + \theta_0(1 - \sum_{i=1}^{I} x_i(\theta)).$$

- The IC condition requires that $X_i(\cdot)$ nondecreasing
- Feasibility requires that $\sum_{i=1}^{I} x_i(\theta) \leq 1$ and $x_i(\theta) \geq 0 \forall i, \theta$.
- The expected transfers associated to the optimal auction are

$$T_i(\theta_i) = E_{\theta_{-i}} t_i(\theta_i, \theta_{-i}) = -\theta_i X_i(\theta_i) + \int_{\underline{\theta}}^{\theta} X_i(\tilde{\theta}_i) d\tilde{\theta}_i.$$

# Solution of optimal auctions

- Let $J_i(\theta_i) = \theta_i - \frac{1-P_i(\theta_i)}{p_i(\theta_i)}$ be the virtual valuation of buyer $i$.
- Ignoring the IC constraint, the optimal auction is $x_i(\theta_i) = 1$ if and only if $J_i(\theta_i) = \max_j J_j(\theta_j)$.
- If the monotone hazard rate condition holds, then $J_i(\theta_i)$ is increasing. If all bidders are symmetric, then $x_i = 1$ if and only if $\theta_i = \max_j \theta_j$:
- In any optimal auction, the object is allocated to the buyer with the highest valuation.

# Solutions of optimal auctions

- The optimal auction is not necessarily efficient. Why? The seller uses his *real valuation* $\theta_0$ to set the ”reserve price” but the *virtual valuations* of the buyers, reflecting incomplete information about their types. Hence it may be that the seller does not sell even though $\max \theta_i \geq \theta_0$.
- All auctions resulting in the same decision (allocating the object to the highest bidder) result in the same revenue: *this is the revenue equivalence theorem*

# Optimal bargaining processes

- In the Myerson-Satterthwaite environment
- We let $w(c, v)$ denote the transfer from buyer to seller

$$X_i(c) \equiv E_v[x(c, v0], X_2(v) \equiv E_c[x(c, v)],$$

$$W_1(c) \equiv E_v[w(c, v)], W_2(v) \equiv E_c[w(c, v)]$$

-

$$U_1(c) = -cX_1(c) + W_1(c), U_2(v) = vX_2(v) - W_2(v)$$

# Optimal bargaining processes

- The mechanism is IR if $U_1(c) \geq 0, U_2(v) \geq 0$ for all $c, v$
- The mechanism satisfies IC if

$$
\begin{aligned}
U_1(c) &\geq -cX_1(\hat{c}) + W_1(\hat{c}) \forall c, \hat{c} \\
U - 2(v) &\geq vX_2(\hat{v}) - W_2(\hat{v}) \forall v, \hat{v}
\end{aligned}
$$

# The planner's objective

- Consider a benevolent planner who wants to maximize expected social surplus $E_{c,v}[(v-c)x(c,v)]$ s.t. Ic, IR and BB constraints.
- We know from the previous computations that IC, IR and Bb imply that

$$E_{c,v}[J_2(v) - J_1(c)]x(c,v) \geq 0,$$

where $J_1(c) = c + \frac{P-1(c)}{p_1(c)}, J_2(v) = v - \frac{1-P_2(v)}{p_2(v)}$ are the virtual surpluses.

# The planner's objective

- Let $\mu$ be the Lagrange multiplier. We write the Lagrangian of the problem:

$$\mathcal{L} = E_{c,v}[(v - c) + \mu[J_2(v) - J_1(c)]x(c, v)$$

- And the first-order condition is

$$x(c, v) = \begin{cases} 1 & \text{if } v + \mu_2 J(v) \geq c + \mu J_1(c) \\ 0 & \text{otherwise} \end{cases}$$

# The solution

- Trade occurs if and only if

$$v - \frac{\mu}{1+\mu}\frac{1-P_2(v)}{p_2(v)} \geq c + \frac{\mu}{1+\mu}\frac{P-1(c)}{p_1(c)}.$$

- We still need to specify $\alpha = \frac{\mu}{1+\mu}$.
- To do this, note that ideally the constraint must be satisfied with equality:

$$E_{c,v}[J_2(v) - J_1(c)]x(c,v) = 0.$$

# Solution with uniform distributions

- If distributions are uniform on $[0,1]$, $P_1(c) = c$ and $P_2(v) = v$ and trade occurs if and only if $v - c \geq \frac{\alpha}{1+\alpha}$. Substituting yields

$$\int_0^{1-\frac{\alpha}{1+\alpha}} [\int_{c+\frac{\alpha}{1+\alpha}}^{1} [2v - 1 - 2c] dvdc = 0,$$

- The solution is $\frac{\alpha}{1+\alpha} = \frac{1}{4}$
- This is the same as in the Chatterjee Samuelson double auction which is thus optimal!

# Correlated types

- What happens if types are correlated?
- The principal can implement the same allocation as if he had information on agent's types
- the idea is to use a "shoot all mechanism":
  - If all agents announce the same information, the principal implements the first best for this announcement
  - If announcements do not coincide, all agents get $-\infty$
- So clearly if all other agents announce the true vector of types, each agent has an incentive to announce the truth as well.

# Cremer and Mac Lean

- Even if there is a small amount of correlation, the principal can implement the first best.
- We use the construction of Cremer and Mac Lean (1985) to show it:
- Agent $i$ can have two types $\underline{\theta}_i$ or $\overline{\theta}_i$
- Let $q_{11}$ and $q_{12}$ be the probability that $\theta_2 = \underline{\theta}_2$ and $\theta_2 = \overline{\theta}_2$when $\theta_1 = \underline{\theta}_1$
- Let $q_{21}$ and $q_{22}$ be defined similarly when $\theta_1 = \overline{\theta}_1$.

# Cremer and Mac Lean

- Suppose that the full rank condition holds (some correlation): $q_{11}q_{22} \neq q_{12}q_{21}$.
- Let $t_{11}$ and $t_{12}$ denote the transfer to agent 1 when he announces $\underline{\theta}_1$ and agent 2 announces $\underline{\theta}_2$ and $\overline{\theta}_2$
- Let $t_{21}$ and $t_{22}$ be defined similarly when agent 1 announces $\overline{\theta}_1$.

# Cremer Mac Lean

- Let

$$\begin{aligned} A_1 &\equiv q_{11}(U^*_{11} - V_1(x^*_{11}, \underline{\theta}_1) + q_{12}(U^*_{12} - V_1(x^*_{12}, \underline{\theta}_1) \\ A_2 &\equiv q_{21}(U^*_{21} - V_1(x^*_{21}, \overline{\theta}_1) + q_{22}(U^*_{22} - V_1(x^*_{22}, \overline{\theta}_1) \end{aligned}$$

- The transfers must satisfy

$$\begin{aligned} q_{11}t_{11} + q_{12}t_{12} &= A_1 \\ q_{21}t_{21} + q_{22}t_{22} &= A_2 \end{aligned}$$

# Cremer Mac Lean

- Similarly define

$$\begin{aligned} A_3 &\equiv q_{11}(V_1(x^*_{21},\underline{\theta}_1)-V_1(x^*_{11},\underline{\theta}_1)) \\ &+ q_{12}(V_1(x^*_{22},\underline{\theta}_1)-V_1(x^*_{12},\underline{\theta}_1)) \\ A_4 &\equiv q_{21}(V_1(x^*_{11},\overline{\theta}_1)-V_1(x^*_{21},\overline{\theta}_1)) \\ &+ q_{22}(V_1(x^*_{12},\overline{\theta}_1)-V_1(x^*_{22},\overline{\theta}_1)) \end{aligned}$$

- The IC imply that

$$\begin{aligned} q_{11}(t_{11}-t_{21})+q_{12}(t_{12}-t_{22}) &\geq A_3 \\ q_{21}(t_{21}-t_{11})+q_{22}(t_{22}-t_{12}) &\geq A_4 \end{aligned}$$

# Cremer Mac Lean

- Substituting yields

$$(q_{11}q_{22} - q_{12}q_{21})t_{11} \geq A_1 q_{22} + (A_4 - A_2)q_{12}$$

$$(q_{11}q_{22} - q_{21}q_{12})t_{21} \leq -A_2 q_{12} - (A_3 - A_1)q_{22}.$$

- So it is always possible to design transfers $t_{21}$ when announcements are different to implement the optimal choice
- However, as types become less correlated, $(q_{11}q_{22} - q_{21}q_{12})$ goes to 0 and transfers have to become arbitrarily large..

# Efficiency at the limit

- Consider the bilateral trade model of Myerson-Satterthwaite
- As the number of agents on the market goes to infinity, efficiency is restored because every agent is ”small”
- In the public good model, this is not necessarily the case and efficiency may not be restored for BB mechanisms even when players become small.

# Final issues

- The model has been extended to study informed principal problems
- Risk aversion and limited liability
- Dynamic interaction with persistent types..

# Summary

- Mechanism with several agents consider either the implementation of efficient allocations or revenue maximization by the principal
- Without budget balance, the Groves-Clarke mechanisms can be used to elicit information about players' types in public decisions
- At the interim stage, the AGV mechanism also satisfies budget balance
- In the Myerson-Satterthwaite bilateral trade model, efficiency cannot be obtained, and one can characterize the best mechanism
- In auctions, optimal mechanisms are characterized and lead to allocate the good to the bidder with the highest valuation