Lecture III: Mechanism Design I

Francis Bloch

[1] Université Paris I

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Mechanism design

- We look at the vast literature on *mechanism design*
- Players are privately informed
- A planner/principal/mechanism designer constructs a game form /mechanism/contract to elicit information about the players' types
- Players send a message about their types to the planner
- Outcomes are realized

Mechanism design and implementation

- The question raised in implementation theory is: *What are the social choice functions (mappings from the set of types to the set of outcomes) that can be implemented using a game form?*
- The question we ask here is different: suppose that the social choice function is known (e.g. efficient outcomes, core outcomes, outcomes which maximize the utility of the principal). *Can we characterize those mechanisms which implement the given social choice function?*

A first example: monopolistic price discrimination

- A monopolist produces a good at constant marginal cost c and want to sell it to a buyer
- The consumer receives utility

$$u_1(q, T, \theta) = \theta V(q) - T,$$

where $\theta V(q)$ is gross surplus $V(0) = 0$, $V' > 0$ and $V'' < 0$ and T is the transfer

- Type θ is privately known by the consumer. The seller only knows that $\overline{\theta} > \underline{\theta}$ occur with probabilities $\overline{p}$ and $\underline{p}$.

Price discrimination

- The seller proposes a nonlinear tariff $T(q)$ (and commits to it)
- The consumer can accept the mechanism choose a consumption q and pay $T(q)$
- or could reject the mechanism
- If the seller knew the player's type, she would offer a fix q and maximize $\theta V(q) - cq$ and it would be maximized at $\theta V'(q) = c$.
- Because the seller does not know the type, she will propose two contracts/ outcomes $(\underline{q}, \underline{T})$ and $(\overline{q}, \overline{T})$ targeted at the low type and high type consumers.
- The seller's profit is then

$$Eu_0 = \underline{p}(\underline{T} - c\underline{q}) + \overline{p}(\overline{T} - \overline{q}).$$

Individual Rationality constraints

- The Individual Rationality (IR) constraints indicate that the consumer wants to participate no matter his type

$$\begin{aligned} \underline{\theta} V(\underline{q}) - \underline{T} &\geq 0 \text{ IR1} \\ \overline{\theta} V(\overline{q}) - \overline{T} &\geq 0 \text{ IR2} \end{aligned}$$

Incentive compatibility constraints

- Incentive compatibility (IC) constraints indicate that the consumer picks the bundle intended for his type

$$\begin{aligned} \underline{\theta} V(\underline{q}) - \underline{T} &\geq \underline{\theta} V(\overline{q}) - \overline{T} \text{ IC1} \\ \overline{\theta} V(\overline{q}) - \overline{T} &\geq \overline{\theta} V(\underline{q}) - \underline{T}. \text{ IC2} \end{aligned}$$

Analysis

- We show that only IR1 and IC2 are binding.
- If IR1 and IC2 are satisfied,

$$\overline{\theta}V(\overline{q}) - \overline{T} \geq (\overline{\theta} - \underline{\theta})V(\underline{q}) \geq 0.$$

- IR 1 must be binding. If not, the seller could increase $\underline{T}$ and $\overline{T}$ by the same amount, keep all constraints satisfied and get higher revenues
- IC2 must be binding. If not, the seller could increase $\overline{T}$ keep all constraints satisfied and get higher revenues.
- We ignore IC1 and check that the solution to the relaxed problem satisfies IC1.

The solution

- Maximizing Eu_0 subject to IR1 and IC2 is equivalent to maximizing

$$\begin{aligned} \underline{p}(\underline{T} - c\underline{q}) + \overline{p}(\overline{T} - c\overline{q}) &= [(\underline{p}\underline{\theta} - \overline{p}(\overline{\theta} - \underline{\theta}) \\ &\quad V(\underline{q}) - \underline{p}c\underline{q}] + \overline{p}(\overline{\theta}V(\overline{q}) - c\overline{q}) \end{aligned}$$

- resulting in

$$\begin{aligned} \underline{\theta}V'(\underline{q}) &= \frac{c}{1 - \frac{\overline{p}(\overline{\theta} - \underline{\theta})}{\underline{p}\underline{\theta}}}, \\ \overline{\theta}V'(\overline{q}) &= c. \end{aligned}$$

Interpretation

- The quantity purchased by high types is socially optimal (no distortion at the top)
- The quantity purchased by low types is socially suboptimal (as $V'' < 0$)
- To check that IC1 is satisfied

$$\underline{\theta} V(\overline{q}) - \overline{T} = -(\overline{\theta} - \underline{\theta})[V(\overline{q} - V(\underline{q})] < 0,$$

while

$$\underline{\theta} V(\underline{q}) - \underline{T} = 0.$$

- We can reinterpret the mechanism by saying: the seller asks the consumer to report his type θ and proposes a bundle $q(\theta), T(\theta)$ as a function of the consumer's type..

Optimal auctions

- We now analyze an example of mechanism design with several agents.
- In the previous lecture, we analyzed optimal play of buyers for different fixed auction formats (first price, second price, all pay..)
- We now consider the more general problem: suppose that the seller can choose the auction format, what would he choose to maximize his revenue?
- A seller sells a unit of a good to two potential buyers who are ex ante identical.
- The valuations θ_1 and θ_2 take value $\underline{\theta}$ with probability $\underline{p}$ and $\overline{\theta}$ with probability $\overline{p}$.
- The seller asks the two buyers to report their values.

Optimal auctions

- Let s_1 and s_2 be the messages sent by the two buyers.
- The seller chooses a probability $x_i(s_1, s_2)$ of allocating the good to seller i
- a transfer $T_i(s_1, s_2)$ from the buyer to the seller.
- In the first and second price auctions, $x_i = 1$ if $s_i > s_j$. In the first price auction $T_i = s_i$ if $s_i > s_j$ whereas in the second price auction $T_i = s_j$ if $s_i > s_j$.

IR and IC constraints

- Let

$$V(\theta_1, s_1, s_2) = E_{\theta_2}[\theta_1 x_1(s_1(\theta_1), s_2(\theta_2)) - T_1(s_1(\theta_1), s_2(\theta_2)]$$

- In the auction setting, the IR constraint is: for all θ_1,

$$V(\theta_1, s_1, s_2) \geq 0.$$

- And the IC constraint is: for all θ_1 and s_1',

$$V(\theta_1, s_1, s_2) \geq V(\theta_1, s_1', s_2).$$

Direct mechanisms and auctions

- We restrict attention to *direct mechanisms* where messages are players' types ($s_i = \hat{\theta}_i$)
- Let $(\hat{\theta}_1, \hat{\theta}_2)$ be the announced types of the two buyers.
- The seller chooses $x_i(\hat{\theta}_1, \hat{\theta}_2)$ and $T_i(\hat{\theta}_1, \hat{\theta}_2)$.
- As buyers do not know each other's type, they use expected utilities to make their decisions.
- Assuming that the other buyer tells the truth, we define

$$X_i(\hat{\theta}_i) = E_{\theta_2} x_i(\hat{\theta}_i, \theta_j), \; T_i(\hat{\theta}_i) = E_{\theta_2} T_i(\hat{\theta}_i, \theta_j).$$

- These are the only relevant quantities for buyer i.

Optimal auction with two types

- With two types, we let $\underline{X}$ and $\overline{X}$ denote the expected probabilities for a player announcing that his type is low (respectively high) with corresponding transfers $\underline{T}$ and $\overline{T}$.
- The IR constraints now write

$$\begin{aligned} \underline{\theta X} - \underline{T} &\geq 0 \text{ IR1} \\ \overline{\theta X} - \overline{T} &\geq 0 \text{ IR2} \end{aligned}$$

- And the IC constraints write

$$\begin{aligned} \underline{\theta X} - \underline{T} &\geq \underline{\theta}\overline{X} - \overline{T} \text{ IC1} \\ \overline{\theta X} - \overline{T} &\geq \overline{\theta}\underline{X} - \underline{T} \text{ IC2} \end{aligned}$$

Optimal auction with two types

- The seller's expected payoff (if the good has zero cost) is

$$Eu_0 = \underline{p}\,\underline{T} + \overline{p}\,\overline{T}.$$

- Building on the intuition for the single agent case suppose that only IR1 and IC2 are binding..
- Then expected payments are $\underline{T} = \underline{\theta}\underline{X}$ and $\overline{T} = \overline{\theta}(\overline{X} - \underline{X}) + \underline{\theta}\underline{X}$.
- And the seller's expected profit is

$$Eu_0 = (\underline{\theta} - \overline{p}\overline{\theta})\underline{X} + \overline{p}\overline{\theta}\overline{X}.$$

Optimal auction with two types

- What is the space in which $\underline{X}$ and $\overline{X}$ are chosen?
- These are nor probabilities in $[0, 1]$ but *expected probabilities.*
- By symmetry, the ex ante expected probability of getting the good is

$$\underline{p}\underline{X} + \overline{p}\overline{X} \leq \frac{1}{2}.(*)$$

- If $\underline{\theta} < \overline{p}\overline{\theta}$, the seller's profit is increasing in $\overline{X}$ and decreasing in $\underline{X}$. The seller sets $\underline{X} = 0$ (does not sell the good to a low type bidder) and $\overline{X}$ as high as possible, i.e. $\overline{X} = \underline{p} + \frac{\overline{p}}{2}$. The seller sells with probability 1 to the buyer if he is the only buyer with high value, and randomizes among the two buyers with high value..

Optimal auction with two types

- Suppose that $\underline{\theta} > \overline{p}\overline{\theta}$. Then the seller's profit is increasing in both $\underline{X}$ and $\overline{X}$. and the constraint (*) must be binding
- Replacing

$$Eu_0 = \frac{1}{2\underline{p}}(\underline{\theta} - \overline{p}\overline{\theta}) + \frac{\overline{p}}{\underline{p}}(\overline{\theta} - \underline{\theta})\overline{X}.$$

- Again, we pick the highest value for $\overline{X}$, $\overline{X} = \underline{p} + \frac{\overline{p}}{2}$.
- But this time we also choose $\underline{X} = \frac{p}{2}$, i.e. the seller chooses to sell to any of the two buyers with equal probability if they both announce low values.

Revenue equivalence

- A famous result in auction theory is the revenue equivalence theorem.
- Here if $\underline{\theta} > \overline{p}\overline{\theta}$, all auctions which allocate the good to the highest bidder result in the same revenue
- If instead $\underline{\theta} < \overline{p}\overline{\theta}$, the optimal auction must prevent low value bidders from winning the object. This can be done by *imposing a reserve price* like $\underline{\theta}$ under which bids will not be cosnidered..

General mechanism design

- In the general abstract model we suppose that there are $I+1$ players
- Player 0 is the principal with no information
- Players $i = 1, .., I$ are agents with types $(\theta_1, .., \theta_I)$ in some set Θ.
- The principal selects two objects: a decision x and a vector of monetary transfers $(t_1, .., t_n)$, $y = (x, t)$ is the allocation
- Agents have von Neumann Morgenstern utility functions: $u_i(y, \theta)$; the principal has utility function $u_0(y, \theta)$.
- The functions u_i are increasing in t_i, the function u_0 is decreasing in t_i.

General mechanism design

- The agent's expected utility (at the interim stage) is

$$U_i(\theta_i) = E_{\theta_{-i}}[u_i(y(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}|\theta_i],$$

- The expected utility of the principal (at the ex ante stage) is

$$E_\theta u_o(y(\theta), \theta).$$

- A *mechanism* defines a message space $\mathcal{M}_i$, where $\mu = (\mu_1, ..., \mu_I)$ is the set of all messages
- The outcome function associates to each profile of messages $(\mu_1, .., \mu_I)$ an outcome $y(\mu)$.

Revelation principle

- The revelation principle (Gibbard (1973), Green Laffont (1977) Myerson (1979)..) states that the principal can do as well with ”direct” mechanisms in which the message space is the type space.
- More complicated mechanisms are not needed..
- Pick a general mechanism μ and let $\mu_i^*(\theta_i)$ denote the equilibrium choice of agent i in the mechanism when his type is θ_i.
- Define anew outcome function from Θ to Y, $\overline{y}$ by

$$\overline{y}(\hat{\theta}) = y(\mu^*(\hat{\theta}).$$

Revelation principle

- If μ^* is a Bayesian equilibrium of the original mechanism, then truth telling, $\hat{\theta} = \theta$ is a Bayesian equilibrium of the new mechanism.

$$\begin{aligned} E_{\theta_{-i}}[u_i(\overline{y}(\theta), \theta_i, \theta_{-i}|\theta_i] &= E_{\theta_{-i}}[u_i(y(\mu^*(\theta), \theta_i, \theta_{-i}|\theta_i] \\ &= \sup_{\mu_i} E_{\theta_{-i}}[u_i(y(\mu^*_{-i}(\theta_{-i}), \mu_i, \theta_i, \theta_{-i}|\theta_i] \\ &= \sup_{\theta_i} E_{\theta_{-i}}[u_i(\overline{y}(\theta_{-i}, \theta_i), \theta_i, \theta_{-i}|\theta_i] \end{aligned}$$

Single agent

- There is a single agent (like in the price discrimination problem)
- Types are distributed in the compact space $[\underline{\theta}, \overline{\theta}]$ according to the distribution $P(\cdot)$ with density $p(\cdot)$
- The type space is single dimensional (important!) and the decision space also unidimensional (for convenience)
- An allocation is a mapping from θ to $y(\theta) = (x(\theta), t(\theta))$.
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Implementable decisions

- A decision function $x : \theta \to x(\theta)$ is *implementable* if there exists a transfer function $t(\theta)$ such that the allocation $y(\theta) = (x(\theta), t(\theta))$ satisfies the incentive compatibility constraint

$$u_1(y(\theta), \theta) \geq u_1(y(\hat{\theta}, \theta) \text{ for all } \theta, \hat{\theta}.$$

Proposition

A decision function is implementable only if

$$\frac{\partial}{\partial \theta} \frac{\frac{\partial u_1}{\partial x}}{\frac{\partial u_1}{\partial t}} \frac{dx}{d\theta} \geq 0.$$

Proof of the proposition

- Let $\Phi(\hat{\theta}, \theta) = u_1(x(\hat{\theta}, t(\hat{\theta}), \theta)$.
- If truth telling is a best response we must have $\frac{\partial \Phi}{\partial \hat{\theta}}|(\theta, \theta) = 0$.
- In addition, the second order condition $\frac{\partial^2 \Phi}{\partial \hat{\theta}^2}(\theta, \theta) \leq 0$.
- After some algebraic manipulations we obtain the condition of the proposition.

The single crossing condition

- The *single crossing* or (Spence Mirrlees) condition is that either $\frac{\partial}{\partial\theta}\frac{\frac{\partial u_1}{\partial x}}{\frac{\partial u_1}{\partial t}} > 0$ or $\frac{\partial}{\partial\theta}\frac{\frac{\partial u_1}{\partial x}}{\frac{\partial u_1}{\partial t}} < 0$.
- Under the single crossing condition, the necessary condition for implementability is that x is monotonic in θ

Proposition

Assume that CS holds. A necessary condition for $x(\cdot)$ to be implementable is that it be nondecreasing, ie $\theta_2 > \theta_1 \Rightarrow x(\theta_2) \geq x(\theta_1)$.

Proposition

Suppose that CS holds and that $\left|\frac{\frac{\partial u_1}{\partial x}}{\frac{\partial u_1}{\partial t}}\right| \leq K_0 + K_1|t|$. *Then any decision function satisfying* $\frac{dx}{d\theta} \geq 0$ *is implementable.*

Optimal mechanisms

- As we have defined the set of implementable decisions, we now compute the optimal decision rule from the point of view of the principal.
- Additional assumptions are needed:
  1. The reservation utility $\underline{u}$ is independent of the agent's type.
  2. Utilities of the principal and of the agent are quasi-linear:

$$
\begin{aligned}
u_0(x, t, \theta) &= V_0(x, \theta) - t, \\
u_1(x, t, \theta) &= V_1(x, \theta) + t
\end{aligned}
$$

The principal's program

- The principal's program is to maximize

$$E_\theta u_0(x(\theta), t(\theta), \theta),$$

- subject to
- IR constraint:

$$u_1(x(\theta), t(\theta), \theta) \geq \underline{u} \equiv 0.$$

- IC constraint

$$u_1(x(\theta), t(\theta), \theta) \geq u_1(x(\hat{\theta}, t(\hat{\theta}), \theta) \text{ for all } \theta, \hat{\theta}.$$

The "Mirrlees' trick"

- We first observe that the IR constraint must be satisfied with equality at $\theta = \underline{\theta}$:

$$u_1(x(\underline{\theta}), t(\underline{\theta}), \underline{\theta}) = 0.$$

- Next write the indirect utility

$$U_1(\theta) = \max_{\hat{\theta}} u_1(x(\hat{\theta}), t(\hat{\theta}), \theta).$$

- By the envelope theorem

$$\frac{dU_1}{d\theta} = \frac{\partial V_1}{\partial \theta}.$$

- So that

$$U_1(\theta) = \int_{\underline{\theta}}^{\theta} \frac{\partial V_1}{\partial \theta}(x(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta}.$$

The principal's problem

- In addition $u_0 = V_0 + V_1 - U_1$ (because of quasi-linearity) so that the principal's objective is

$$\int_{\underline{\theta}}^{\overline{\theta}} [V_0(x(\theta), \theta) + V_1(x(\theta), \theta) - \int_{\underline{\theta}}^{\theta} \frac{\partial V_1}{\partial \theta}(x(\tilde{\theta}, \tilde{\theta}))d\tilde{\theta}]p(\theta)d\theta.$$

- Integrating by parts:

$$\int_{\underline{\theta}}^{\overline{\theta}} [V_0(x(\theta), \theta) + V_1(x(\theta), \theta) - \frac{1 - P(\theta)}{p(\theta)} \frac{\partial V_1}{\partial \theta}(x(\theta), \theta)]p(\theta)d\theta.$$

The principal's problem

- Next recall that the IC constraint is equivalent to the fact that $x(\theta)$ is non-decreasing.
- So the principal's problem is to maximize

$$\int_{\underline{\theta}}^{\overline{\theta}} [V_0(x(\theta),\theta)+V_1(x(\theta),\theta)-\frac{1-P(\theta)}{p(\theta)}\frac{\partial V_1}{\partial\theta}(x(\theta),\theta))]p(\theta)d\theta.$$

- subject to the constraint that $x(\theta)$ is non-decreasing.
- We then compute the agent's utility

$$U_1(\theta) = \int_{\underline{\theta}}^{\theta} \frac{\partial V_1}{\partial\theta}(x(\tilde{\theta}),\tilde{\theta})d\tilde{\theta}.$$

- to obtain the transfer

$$t(\theta) = U_1(\theta) - V_1(x(\theta),\theta).$$

The solution

- Ignoring the monotonicity constraint, we compute the optimal solution of the relaxed problem by pointwise maximization

$$\frac{\partial V_0}{\partial x} + \frac{\partial V_1}{\partial x} = \frac{1 - P(\theta)}{p(\theta} \frac{\partial^2 V_1}{\partial x \partial \theta}.$$

- Notice that $V_1 + V_0$ is total surplus, so maximizing total surplus would amount to $\frac{\partial V_0}{\partial x} + \frac{\partial V_1}{\partial x} = 0$
- We are adding another term capturing the information rents of the agent..
- At $\theta = \overline{\theta}$, $1 - P(\overline{\theta}) = 0$, *no distortion at the top*

Suboptimal quantities

- The quantity chosen is lower than the socially optimal quantity. Let $\hat{x}(\theta)$ be the socially optimal quantity and $x^*(\theta)$ the solution.
- We have

$$V_0(\hat{x}(\theta), \theta) + V_1(\hat{x}(\theta), \theta) \geq V_0(x^*(\theta), \theta) + V_1(x^*(\theta), \theta),$$

- and

$$V_0(x^*(\theta), \theta) + V_1(x^*(\theta), \theta) - \frac{1-P}{p}\frac{\partial V_1}{\partial \theta}(x^*(\theta), \theta) \geq$$
$$V_0(\hat{x}(\theta), \theta) + V_1(\hat{x}(\theta), \theta) - \frac{1-P}{p}\frac{\partial V_1}{\partial \theta}(\hat{x}(\theta), \theta)$$

- Adding up and using the single crossing condition $x^*(\theta) \leq \hat{x}(\theta)$.

Virtual surplus and monotone hazard rate

- Following Myerson (1981), we define the agent's *virtual surplus* as

$$V_1(x,\theta) - \frac{1-P(\theta)}{p(\theta)} \frac{\partial V_1}{\partial \theta}(x,\theta).$$

- Replacing the agent's surplus by his virtual surplus gives the optimal solution..
- A sufficient condition for $x^*(\theta)$ to be increasing is that *the hazard rate* $\frac{p(\theta)}{1-P(\theta)}$ *is increasing.*

Summary

- When agents have private information, the principal or mechanism designer chooses a game form to elicit information from the agents.
- Mechanism design can occur with a single agent (principal-agent model) or with multiple agents
- By the revelation principle, we can restrict attention to direct mechanisms where agents report their types.
- We compute the solution to the principal agent's problem by using a series of steps (Mirrlees' trick, integration by parts, pointwise maximization)
- There is no distortion at the top and under the monotone hazard rate condition, the decision function is monotonic in types.