# Lecture II: Bayesian Games

Francis Bloch

[1] Université Paris I

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# Strategic interaction and games

- We analyzed the value of information in two cases:(i) individual decision-making and (ii) competitive equiiibrium
- We now look at the richer environment where a small number of decision makers interact
- These strategic situations are represented by *games*
- A simple normal form game is defined by
  - A set $N$ of players
  - A strategy space $S_i$ for each player
  - A utility function: $u_i : \times_{i \in N} S_i \rightarrow \Re$ for each player

# Imperfect and incomplete information

- A games has *imperfect information* if players do not observe the strategies of the other players (e.g. simultaneous games)
- A game has *incomplete information* if some element of the game (most often payoff vectors) are uncertain
- Informally, in a game with incomplete information, every payer has a *type*, which may be known or unknown by the other players and payoffs depend on the *profile of type* of all the players..

# A first example: entry with unknown costs

- Entry game with two firms: an incumbent (player 1) with unknown cost and a potential entrant (player 2)
- Player 1's cost can either be 0 or 3. Player 1 knows it but not player 2
- Player 1 can choose either to build or not ; player 2 can choose either to enter or not.

# Payoffs

- if building cost is 3

| | enter | don't |
|---|---|---|
| build | $0, -1$ | $2, 0$ |
| don't build | $2, 1$ | $3, 0$ |

- if building cost is 0

| | enter | don't |
|---|---|---|
| build | $3, -1$ | $5, 0$ |
| don't build | $2, 1$ | $3, 0$ |

# Analysis of the game

- Player 1 has a dominant strategy: build if cost is low, don't build if cost is high
- Player 2 should enter if player 1 does not build and not enter if player 1 builds
- Then if $p_1$ is the probability that the cost is high, player 2 enters if $p_1 > \frac{1}{2}$ and stays out if $p_1 < \frac{1}{2}$.
- Next suppose that the low cost is 1.5 instead of 0..

# Payoffs

- if building cost is 3

| | enter | don't |
|---|---|---|
| build | $0, -1$ | $2, 0$ |
| don't build | $2, 1$ | $3, 0$ |

- if building cost is 1.5

| | enter | don't |
|---|---|---|
| build | $1.5, -1$ | $3.5, 0$ |
| don't build | $2, 1$ | $3, 0$ |

# Analysis of the game

- The analysis is more complex, because building is no longer a dominant strategy when the cost is low. It depends on the behavior of player 2.
- Let $y$ be the probability that 2 enters and $x$ the probability that 1 builds
- The best response of player 2 is enter if $x < \frac{1}{2(1-p_1)}$, not enter if $x > \frac{1}{2(1-p_1)}$ and indifferent if $x = \frac{1}{2(1-p_1)}$
- The best response of player 1 is build if $y < \frac{1}{2}$ and not build if $y > \frac{1}{2}$
- $x - 0, y = 1$ is an equilibrium for all $p$
- $x = 1, y = 0$ is an equilibrium if $p_1 \leq \frac{1}{2}$
- there is also a mixed strategy equilibrium $x = \frac{1}{2(1-p_1)}, y = \frac{1}{2}$ if $p \leq \frac{1}{2}$

# Providing a public good under incomplete information

- There are two players $i = 1, 2$ who decide simultaneously whether to contribute a public good (a binary decision)
- Each player derives a benefit of 1 id the public good is provided but suffers a cost of $c_i$
- The costs are identically, independently distributed according to a distribution $P(\cdot)$ on $[\underline{c}, \overline{c}]$ where $\underline{c} < 1 < \overline{c}$.
- Payoffs are

| | contribute | not |
|---|---|---|
| contribute | $1 - c_1, 1 - c_2$ | $1 - c_1, 1$ |
| don't | $1, 1 - c_2$ | $0, 0$ |

# Strategies and Bayesian equilibrium

- A pure strategy $s_i$ is a *function* from $[\underline{c}, \overline{c}]$ to $\{0, 1\}$ which says whether you contribute *as a function of your private cost*
- Player *i*'s payoff is

$$u_i(s_i, s_j, c_i) = \max\{s_1, s_2\} - c_i s_i.$$

- A *Bayesian equilibrium* is a pair of strategies $s_1^*, s_2^*$ such that $s_i^*$ maximizes

$$E_j c_j u_i(s_i^*, s_j^*(c_j), c_i)'$$

# Analysis of the game

- Let $z_j \equiv \Pr[s_j^*(c_j) = 1]$
- Then $s_i^*(c_i) = 1$ if $c_i < 1 - z_j$ and $s_i^*(c_i) = 0$ if $c_i > 1 - z_j$
- The equilibrium strategy will be a threshold strategy: contribute if $c_i < c^*$ and not contribute if $c_i > c^*$.
- The cutoff must satisfy

$$c_i^* = 1 - P(c_j^*), c_j^* = 1 - P(c_i^*).$$

- Or $c_1^* = c_2^* = c^*$ which satisfies

$$c^* = 1 - P(1 - P(c^*)).$$

- If $P$ is uniform on $[0, 2]$, the equation is $c^* = 1 - \frac{1}{2}(1 - \frac{c^*}{2})$, so $c^* = \frac{2}{3}$
- So if $c \in [\frac{2}{3}, 1]$, the player does not contribute even though the benefit is higher than the cost..

# Types

- A type is not only a private cost, but also the private information of the player, including his beliefs about other players' types..
- Types $(\theta_1, ..\theta_n)$ are drawn from a joint distribution $p$ over $\Theta_1 \times .. \times \Theta_n$
- Types can be independent: $p(\theta_i|\theta_{-i}) = p(\theta_i|\theta'_{-i})$ or correlated

# Strategies

- Strategies are *mappings* from types to actions $A_i$
- A mixed strategy $\sigma_i$ is a mapping from $\Theta_i$ to $\Delta(A_i)$
- In the *private values* case the utility of player $i$ only depends on his own type: $u_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i)$
- In that case the type of player $j \neq i$ only indirectly affects the payoff of player $i$ through his choice of action $s_j(\theta_j)$
- In general cases, the utility of player $i$ may also depend on the types of the other players, $u_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$

# Expected utilities

- Players evaluate their payoffs at the *interim stage*, knowing their type nut ignoring the type of the other players.
- Other possibilities: players evaluate payoffs at the *ex ante* stage before knowing their types or at the *ex post* stage knowing the types of others..
- Hence they use as evaluation criterion

$$E_{\theta_{-i}} u_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}).$$

# Bayesian equilibrium

- A (pure) *Bayesian equilibrium* is a profile of strategies $(s_1^*, .., s_n^*)$ such that for every player $i$,

$$E_{\theta_{-i}} u_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i}) \geq E_{\theta_{-i}} u_i(s_i(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i})$$

  for any strategy $s_i(\theta_i)$ of player $i$.
- Note: the definition could also use *ex ante* evaluations

$$E_{\theta_i, \theta_{-i}} u_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i}) \geq E_{\theta_i, \theta_{-i}} u_i(s_i(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i})$$

- because strategies are defined as functions of the types..
- Bayesian equilibrium (in mixed strategies) always exist in finite games by the same argument as the argument for existence of a Nash equilibrium in pure strategies

# Auctions

- Auctions are a very common way to allocate objects (used for example to sell any publicly owned good: government debt, real estate owned by the government, radio spectrum, drilling rights, etc..
- Distinction between *private values* auctions (like works of art) and *common value auctions* (like drilling rights)
- Distinction between ascending ("English") and descending ("Dutch") auctions
- In all auctions, the object is allocated to the highest bidder
- In *first-price auctions*, the bidder pays his bid (like descending auctions)
- In *second-price auctions*, the bidder pays the bid of the second bidder (like ascending auctions)

# Second price auctions

- Second-price (or Vickrey) auctions are very special and result in a striking outcome!
- Let $v_1$ and $v_2$ be the values of the two bidders taken from a distribution $P(\cdot)$
- If the two bidders bid $b_1$ and $b_2$, the payoff is

$$u_1(b_1, b_2, v_1) = \begin{cases} v_1 - b_2 \text{ if } b_1 > b_2 \\ \frac{v_1 - b}{2} \text{ if } b_1 = b_2 = b \\ 0 \text{ if } b_1 < b_2 \end{cases}$$

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# Equilibrium of a second price auction

- *In a second-price auction, every player has a dominant strategy, which is to bid exactly his value, $b_i = v_i$.*
- *The bidder has no incentive to choose a bid $\hat{b}_i < v_i$.*

| | bid $\hat{b}_i < v_i$ | bid $v_i$ |
|---|---|---|
| $\hat{b}_i < v_i < b_j$ | 0 | 0 |
| $\hat{b}_i < b_j < v_i$ | 0 | $v_i - b_j$ |
| $b_j < \hat{b}_i < v_i$ | $v_i - b_j$ | $v_i - b_j$ |

# Equilibrium of a second-price auction

- *The bidder has no incentive to choose a bid $\hat{b}_i > v_i$.*

| | bid $\hat{b}_i > v_i$ | bid $v_i$ |
|---|---|---|
| $v_i < \hat{b}_i < b_j$ | 0 | 0 |
| $v_i < b_j \hat{b}_i$ | $v_i - b_j$ | 0 |
| $b_j < v_i < \hat{b}_i$ | $v_i - b_j$ | $v_i - b_j$ |

# First-price auctions with discrete types

- We first consider a *discrete* auction. Each bidder can have any of two values: $\underline{\theta}$ (with probability $\underline{p}$) and $\overline{\theta}$ (with probability $\overline{p}$)
- The valuations are independently distributed.
- *Players must be playing a mixed strategy equilibrium:* If one player plays a pure strategy and bids $\underline{b}$ and $\overline{b}$ , the other player has an incentive to bid slightly higher $\underline{b} + \epsilon$ and $\overline{b} + \epsilon$..
- We look for an equilibrium where bidder bid $\underline{\theta}$ is their type is $\underline{\theta}$ and plays a mixed strategy $F$ over $[\underline{b}, \overline{b}]$ if their type is $\overline{\theta}$
- Suppose that $\underline{s} = \underline{\theta}$.

# First-price auctions with discrete types: Equilibrium

- In equilibrium, a player must be indifferent among all strategies in the support:

$$(\overline{\theta} - s)[\underline{p} + \overline{p}F(s)] = cste$$

- And as $F(\underline{\theta}) = 0$ when $s = \underline{s} = \underline{\theta}$,

$$(\overline{\theta} - s)[\underline{p} + \overline{p}F(s)] = (\overline{\theta} - \underline{\theta})\underline{p}.$$

- At the upper end of the support, $\overline{s}$, $F(\overline{s}) = 1$ so that

$$(\overline{\theta} - \overline{s}) = (\overline{\theta} - \underline{\theta})\underline{p}.$$

# Social surplus, utilities and revenues in the first-price auction with discrete types

- The expected social surplus (ex ante) is $\underline{p}^2\underline{\theta} + (1 - \underline{p}^2)\overline{\theta}$
- Bidder's utilities are 0 if the type is *underline*$\theta$ and $\underline{p}(\overline{\theta} - \underline{\theta})$ if the type is $\overline{\theta}$
- Expected social surplus and bidders' utilities are the same as in a second-price auction
- Hence seller's expected revenues (which is equal to expected social surplus minus expected bidders utilities) is the same in the first-price and second-price auction with discrete types. This is one instance of the *revenue equivalence theorem*.

# First price auction with a continuum of types

- Two bidders $i = 1, 2$ with valuations drawn from the same distribution $P$ with density $p$ on $[\underline{\theta}, \overline{\theta}]$.
- Payoffs are

$$u_1(b_1, b_2, v_1) = \begin{cases} cv_1 - b_1 \text{ if } b_1 > b_2 \\ \frac{v_1 - b}{2} \text{ if } b_1 = b_2 = b \\ 0 \text{ if } b_1 < b_2 \end{cases}$$

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- The expected utility of a bidder is

$$E_{b_2} u_1(b_1, b_2, v_1) = (v_1 - b_1) \Pr(b_1 > b_2).$$

# Monotonicity of bids

- Suppose that $v_1 > v_1'$. We show that $b_i \geq b_i'$
- Because $b_1$ is an optimal strategy when the value is $v_1$:

$$(v_1 - b_1)\Pr[b_1 > b_2] \geq (v_1 - b_1')\Pr[b_1' > b_2],$$

- Because $b_1'$ is an optimal strategy when the value is $v_1'$:

$$(v_1' - b_1')\Pr[b_1' > b_2] \geq (v_1' - b_1)\Pr[b_1 > b_2],$$

- Adding up and simplifying

$$(v_1 - v_1')(\Pr[b_1 > b_2] - \Pr[b_1' > b_2]) \geq 0.$$

- So if $v_1 > v_1'$ then $b_1 \geq b_1'$.

# No atoms or gaps

- *The bidding functions cannot have atoms*: If $b_j$ has an atom at $s$, then $i$ should assign 0 probability at bidding between $s - \epsilon$ and $s$, as it does better by bidding just above $s$. But then $j$ should lower its atom to $s - \epsilon$..
- *The bidding functions must be continuous*: If $i$ does not bid in $[s_i^-, s_i^+]$ then $j$ should not bid in the same interval. But then $i$ has an incentive to lower his bid from $s_i^+$ to $s_i^- + \epsilon$..
- Hence the bidding functions $b_i(v_i)$ must be continuous and strictly increasing.

# Computing the equilibrium strategies

- Because the bidding function is strictly increasing, there is a one-to-one relation between the value and the bid.
- Instead of assuming that players choose bids given their value, suppose that there is a fixed bidding function $b_i(\cdot)$ and that bidders "claim" to have a value $\hat{v}_i$ resulting in the bid $b_i(\hat{v}_i)$.
- Let $U(v_1, \hat{v}_1)$ be the expected utility of player 1 when her true value is $v_1$ and she claims to have value $\hat{v}_1$ and bids $b_1(\hat{v}_1)$.
- We compute

$$U(v_1, \hat{v}_1) = (v_1 - b_1(\hat{v}_1)) \Pr[b_1(\hat{v}_1) > b_2(v_2)].$$

# The equilibrium strategies

- Suppose that the two players play *symmetric strategies*, $b_1(\cdot) = b_2(\cdot)$.
- At equilibrium, player 2 plays $b(v_2)$. So the expected utility becomes

$$U(v_1, \hat{v}_1) = (v_1 - b(\hat{v}_1)) \Pr[b(\hat{v}_1) > b(v_2)].$$

- Because $b(\cdot)$ is strictly monotonic, $\Pr[b(\hat{v}_1) > b(v_2)] = \Pr[\hat{v}_1 > v_2] = P(\hat{v}_1)$.
- So

$$U(v_1, \hat{v}_1) = (v_1 - b(\hat{v}_1))P(\hat{v}_1).$$

- In equilibrium, player 1 must maximize $U(v_1, \hat{v}_1)$ in $\hat{v}_1$ at the point $v_1$ so:

$$-b'(v_1)P(v_1) + (v_1 - b(v_1))p(v_1) = 0.$$

# Equilibrium in a first price auction

- The equilibrium of the first-price auction is the solution to the differential equation

$$-b'(v_1)P(v_1) + (v_1 - b(v_1))p(v_1) = 0.$$

with the boundary condition $b(\underline{v}) = 0$.

- If the distribution $P$ is uniform over $[0, 1]$, the equation becomes

$$-b'(v)v + v - b(v) = 0.$$

- This equation has a solution $b(v) = \frac{v}{2}$.

# Revenue equivalence in auctions with a continuum of types

- Suppose that $P(\cdot)$ is uniform over $[0, 1]$
- *In the second price auction:* $b(v) = v$ and the expected revenue of the seller is $E\min\{b_1, b_2\} = E\min\{v_1, v_2\} = \frac{1}{3}$.
- *In the first-price auction:* $b(v) = \frac{v}{2}$ and the expected revenue of the seller is $E = \max\{b_1, b_2\} = E\max\{\frac{v_1}{2}, \frac{v_2}{2}\} = \frac{1}{3}$.
- The expected revenue of the seller is the same in first-price and second-price auctions.

# War of attrition

- A war of attrition is a situation where two players compete to see which is the first to quit the game.
- The player who stays longest wins the prize
- Wars of attrition occur in animal behavior (fighting over a territory), human behavior (see who stays the longest), interaction among firms (wait for another firm to exit an industry..)
- Formally, a war of attrition is like a second price auction where both the winner and the loser pay (this is called an ”all-pay auction”)

# War of attrition

- Suppose that players have a benefit from surviving the war of attrition, $\theta_i$ which is privately known.
- The values $\theta_i$ are distributed independently according to some distribution $P(\cdot)$
- Each player $i, j$ chooses a time to exit $s_i$ as a function of $\theta_i$.
- Payoffs are

$$u_i(s_i, s_j, \theta_i) = \begin{cases} c - s_i \text{ iff } s_i < s_j, \\ \theta_i - s_j \text{ if } s_i > s_j, \\ \frac{\theta_i - s}{2} \text{ if } s_i = s_j = s \end{cases}$$

# Monotonicity of payoffs

- Expected payoffs are

$$U(s_i, \theta_i) = -s_i \Pr[s_i < s_j] + \int_{\theta_j | s_j < s_i} (\theta_i - s_j) p(\theta_j) d\theta_j$$

- Consider two values $\theta_i > \theta_i'$, then $s_i \geq s_i'$.
- We use $U(s_i, \theta_i) \geq U(s_i', \theta_i)$ and $U(s_i', \theta_i') \geq U(s_i, \theta_i')$ to obtain:

$$\int_{\theta_j | s_j < s_i} (\theta_i - \theta_i') p(\theta_j) d\theta_j \geq \int_{\theta_j | s_j < s_i'} (\theta_i - \theta_i') p(\theta_j) d\theta_j \geq 0,$$

- so that $s_i \geq s_i'$
- We can also, following the same steps as for first price auctions, show that the strategy is continuous and strictly increasing in $\theta_i$.

# Computing equilibrium

- As before let $U(\theta_i, \hat{\theta}_i)$ denote the utility of player $i$ when his type is $\theta_i$ and he claims a type $\hat{\theta}_i$.
- Let $s(\cdot)$ be the symmetric strategy of the two players.
- $$U(\theta_i, \hat{\theta}_i) = -s(\hat{\theta}_i)[1 - P(\hat{\theta}_i)] + \int_0^{\hat{\theta}_i} (\theta_i - s(\theta_j))p(\theta_j)d\theta_j.$$
- Taking derivatives with respect to $\hat{\theta}_i$ and computing the derivative at $\theta_i$:

$$-s'(\theta_i)(1 - P(\theta_i)) + p(\theta_i)s(\theta_i) + (\theta_i - s(\theta_i))p(\theta_i) = 0.$$

- resulting in the differential equation

$$-s'(\theta)(1 - P(\theta)) + \theta p(\theta) = 0.$$

# War of attrition with exponential distribution

- If the distribution of values is exponential, $P(x) = 1 - \exp -x$, $p(x) = \exp -x$.
- The differential equation becomes $s'(x) = x$ with the boundary condition $s(0) = 0$.
- And the solution is $s(x) = \frac{x^2}{2}$.

# Double auction

- In a double auction, both the buyer and seller have private information (about the cost and value of a good)
- They simultaneously make offers and the auctioneer chooses a price $p$ to clear the market.
- In the version studied by Chatterjee and Samuelson (1983), the seller has a cost $c$ and the buyer a valuation $v$ in the interval $[0, 1]$
- The seller and buyer simultaneously choose bids $b_1$ and $b_2$
- If $b_1 \leq b_2$,t he two parties trade at a price $p = \frac{b_1+b_2}{2}$.
- If $b_1 > b_2$, the two parties do not trade.

# Double auction

- The seller's utility is

$$u_1 = \begin{cases} fracb_1 + b_2 2 - c \text{ if } b_1 < b_2 \\ 0 \text{ if } b_1 > b_2 \end{cases}$$

- The buyer's utility is

$$u_2 = \begin{cases} cv - \frac{b_1 + b_2}{2} \text{ if } b_1 < b_2 \\ 0 \text{ if } b_1 > b_2 \end{cases}$$

- With symmetric information this is the *Nash demand game*. If $v > c$, this game has a continuum of symmetric equilibria where the two players announce the same $t \in [c, v]$.

# Double auction

- Next assume that information is not symmetric. The distribution of costs is $P_1 \in [0,1]$ and the distribution of values $P_2 \in [0,1]$.
- We look for a pure strategy equilibrium, $s_1(c)$ and $s_2(v)$
- Let $F_1(b) = \Pr[s_1(c) \leq b]$ and $F_2(b) = \Pr[s_2(v) \leq b]$ be the cumulative distributions of seller and buyer bids.
- Expected utility of seller is

$$U_1(b_1, c) = \int_{b_1}^{1} [\frac{b_1 + b_2}{2} - c] f_2(b_2) db_2,$$

- Expected utility of buyer is

$$U_2(b_2, v) = \int_{0}^{b_2} [v - \frac{b_1 + b_2}{2}] f_1(b_1) db_1$$

# Monotonicity of bids

- Using the standard argument, equilibrium bids are increasing in $c$ and $v$
- We focus attention on continuous, strictly increasing bidding strategies.

# Computation of equilibrium

- We compute equilibrium *directly*. Given the distribution of bids $b_2$, the seller chooses $b_1$ to maximize $U_1(b_1, c)$:

$$\frac{1}{2}[1 - F_2(s_1(c))] - (s_1(c) - c)f_2(s_1(c)) = 0.$$

- Similarly the buyer chooses $b_2$ to maximize $U_2(b_2, v)$:

$$\frac{1}{2}F_1(s_2(v)) - [v - s_2(v)]f_1(s_2(v)) = 0.$$

# Equilibrium with uniform distribution

- Chatterjee and Samuelson (1983) suppose that the distributions $P_1$ and $P_2$ are uniform on $[0, 1]$
- They look for linear strategies:

$$\begin{aligned} s_1(c) &= \alpha_1 + \beta_1 c, \\ s_2(v) &= \alpha_2 + \beta_2 v, \end{aligned}$$

- Then

$$F_i(b) = P_i(s_i^{-1}(b)) = s_i^{-1}(b) = \frac{b - \alpha_i}{\beta_i},$$

and

$$f_i(b) = \frac{1}{\beta_i}$$

# Equilibrium

- The first order conditions give then

$$\frac{2(\alpha_1 + (\beta_1 - 1)c}{\beta_2} = \frac{\beta_2 - (\alpha_1 + \beta_1 c) + \alpha_2}{\beta_2},$$
$$\frac{2(1-\beta_2)v - \alpha_2}{\beta_1} = \frac{\alpha_2 + \beta_2 v - \alpha_1}{\beta_1}$$

- We identify the coefficients for all constants, $c$ and $v$ to obtain

$$2(\beta_1 - 1) = -\beta_1, \qquad 2(1-\beta_2) = \beta_2,$$
$$2\alpha_1 = \beta_2 - \alpha_1 + \alpha_2, \qquad -2\alpha_2 = \alpha_2 - \alpha_1,$$

# Equilibrium of the double auction

- We obtain $\beta_1 = \beta_2 = \frac{2}{3}, \alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{12}$
- The players trade if and only if $\alpha_2 + \beta_2 v \geq \alpha_1 + \beta_1 c$, i.e. $v \geq c + \frac{1}{4}$
- So the equilibrium is inefficient: when the difference between $v$ and $c$ is too small, the buyer and seller do not trade..
- There are other equilibria as well: one where $b_1 = 1$ and $b_2 = 0$ (no serious trade), and one where the sellers and buyers trade at a fix price $b$ if $v > b > c$..

# Summary

- Games are multi-agent situations where agents interact strategically
- Games with incomplete information are games where players have possibly different information on the game
- Games with incomplete information are modeled by assuming that players have types, strategies conditional on types and are expected utility maximizers.
- The central solution concept is Bayesian equiibrium
- Bayesian equilibrium is used to solve auctions and related games (like the war of attrition or double auctions..)